Ch.2 #29. Let $G$ be a finite group. Show that the number of elements $x \in G$ such that $x^3 = e$ is odd. Show that the number of elements $x \in G$ such that $x^2 \neq e$ is even.

Proof. Here we are again using only the information given (which doesn’t look like much does it?) but will nonetheless prove the statements. Actually, I will give the proof for the first statement, the argument for the other one is similar.

Let $T = \{ x \in G | x^3 = e \}$.

Observe that $e \in T$ since certainly $e^3 = e$. Now if $x \in T$, then $x^3 = e$ but then $x^{-1}$ also belongs to $T$, to wit:

\[
(x^{-1})^3 = x^{-3} = (x^3)^{-1} = e^{-1} = e \quad \text{think!}
\]

So, for each $x \in T$, we have $x^{-1} \in T$. The question is then, how can we show that $|T|$ is odd? We note the following key fact. FACT: If $x \in T$ and $x \neq e$ then $x \neq x^{-1}$. (Why?) Well, if $x \in T$ ($x \neq e$) and $x^{-1} = x$ then

\[
x^{-1} = x
\]

\[
x^2x^{-1} = x^2x
\]

\[
x = x^3
\]

\[
= e \quad \text{since } x \in T
\]

which contradicts the choice of $x \neq e$. $\rightarrow$
So the elements in $T$ other than the identity come in pairs, i.e. $(x_1, x_1^{-1}), (x_2, x_2^{-1}), \ldots$ etc. where $x_i \neq x_i^{-1}$ and so there are an even number of non identity elements in $T$ and so, if you include $e$ you get an odd number overall.

[Note: The possibility that $|T| = 1$ is also included. After all, if there are no other elements in $T$ then $|T|$ will be odd (1) automatically.]

□