Show that $Z(S_n) = \{i\}$ for $n \geq 3$

Recall that $Z(G) = \{x \in G | gx = xg \ \forall g \in G\}$. As such, to show that $Z(S_n)$ is trivial, it suffices to show that for all $\sigma \neq i \in S_n$, there exists $\tau \in S_n$ such that $\sigma \tau \neq \tau \sigma$.

This being the case, can we easily find a $\tau$ for every $\sigma$? The answer is yes, keeping in mind the fact that for $\delta, \gamma$ in $S_n$, $\delta = \gamma$ if and only if $\delta(x) = \gamma(x)$ for all $x \in X$. Therefore, using the reverse of this:

$$\sigma \tau \neq \tau \sigma \iff \exists x \in X \text{ s.t. } \sigma \tau(x) \neq \tau \sigma(x)$$

So let $\sigma \in S_n$ be a non identity element and say $\sigma(a) = b$ where $a \neq b$ then we can choose $\tau = (bc)$ where $c \neq \sigma(b)$, which is possible since $n \geq 3$.

$$\tau \sigma(a) = \tau(b) = c$$

while

$$\sigma \tau(a) = \sigma(a) = b$$

and since $c \neq \sigma(b)$ then we are done, because we’ve demonstrated that $\sigma \tau$ and $\tau \sigma$ act differently on $a \in X$ and so are different elements of $S_n$. As such, no nonidentity element of $S_n$ commutes with all of $S_n$ and so $Z(S_n) = \{i\}$. 