We are going to consider the Riemann Sums which arise in the determination of the area under the graph of \( y=x^2 \) over the interval \([0,2] \).

We will compare how the approximations depend on not only the number of rectangles ‘n’ but also on whether we use the left, middle or right endpoints of each sub-interval.

```maple
with(student);

rightbox(x^2, x=1..3, 10);
```
\[ \text{rightsum}(x^2, x=1..3, 10) \text{: evalf}(); \]

\[ 9.480000000 \]

\[ \text{leftbox}(x^2, x=1..3, 10); \]
\[ \text{leftsum}(x^2, x=1..3, 10): \text{evalf}(\%) \]

\[ 7.880000000 \]

\[ \text{middlebox}(x^2, x=1..3, 10); \]
\begin{verbatim}
> middlesum(x^2, x=1..3, 10): evalf(%);
8.660000000

> n:=20;leftbox(x^2, x=1..3, n); evalf(leftsum(x^2, x=1..3, n));middlebox
  (x^2, x=1..3, n); evalf(middlesum(x^2, x=1..3, n));rightbox(x^2, x=1..3,
  n); evalf(rightsum(x^2, x=1..3, n));

n := 20
\end{verbatim}
Now let's compare what happens with n=50 rectangles!

\[
\begin{align*}
n &:= 50 \\
\text{leftbox}(x^2, x=1..3, n); \text{evalf(leftsum}(x^2, x=1..3, n)); \\
\text{middlebox}(x^2, x=1..3, n); \text{evalf(middlesum}(x^2, x=1..3, n)); \\
\text{rightbox}(x^2, x=1..3, n); \text{evalf(rightsum}(x^2, x=1..3, n));
\end{align*}
\]

\[n := 50\]
Notice how in each case, the left sum is less than the middle sum which is, in turn, less than the right sum.

It seems, however, that the middle sum seems to cover the region the best.

Note also how we seem to be closing in on a value of 2.6666... for the area.

Indeed, as we shall see, this is, in fact, the exact value.

If you’re still not convinced, consider what happens when we use...
n = 100 rectangles.

> n := 100; leftbox(x^2, x=1..3, n); evalf(leftsum(x^2, x=1..3, n)); middlebox(x^2, x=1..3, n); evalf(middlesum(x^2, x=1..3, n)); rightbox(x^2, x=1..3, n); evalf(rightsum(x^2, x=1..3, n));

n := 100

8.586800000
8.746800000

> # Notice now how the three values start to come together toward
> the middle and how the collections
> # of rectangles in each case seem to more ‘neatly’ cover the
> region under the graph.
> #
> # The exact value of the area is this entity known as the definite
> integral of f(x) over
> # the interval [1,3]
> > Int(x^2,x=1..3);
\begin{align*}
\int_{1}^{3} x^2 \, dx & \quad \text{whose value is} \\
> \text{evalf(int(x^2, x=1..3));} \quad \text{8.66666667}
\end{align*}