Show that $\mathbb{Q}(\sqrt{2})$ is not ring isomorphic to $\mathbb{Q}(\sqrt{5})$.

The first comment to make is that you can’t choose a homomorphism and show that it is not an isomorphism. The question is basically asking you to show that no homomorphism from $\mathbb{Q}(\sqrt{2})$ to $\mathbb{Q}(\sqrt{5})$ can be an isomorphism.

Suppose now that $\phi : \mathbb{Q}(\sqrt{2}) \rightarrow \mathbb{Q}(\sqrt{5})$ is a homomorphism. If $\phi$ is onto then we know that $\phi(1) = 1$ and consequently $\phi(n) = n$ for all $n \in \mathbb{Z}$, and moreover that $\phi(a) = a$ for all $a \in \mathbb{Q}$. Therefore, if $a + b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ then

$$
\phi(a + b\sqrt{2}) = \phi(a) + \phi(b\sqrt{2}) \\
= \phi(a) + \phi(b)\phi(\sqrt{2}) \\
= a + b\phi(\sqrt{2})
$$

So the essential detail to resolve is, what is $\phi(\sqrt{2})$?

If $\phi(\sqrt{2}) = c + d\sqrt{5}$ then what are the possible values for $c$ and $d$? Here we use a tactic similar to that for showing that no non-trivial homomorphism exists from $2\mathbb{Z}$ to $3\mathbb{Z}$. Observe that $\phi(2) = 2$ since $2 \in \mathbb{Z}$, but since $2 = (\sqrt{2})^2$, then $\phi(2) = \phi((\sqrt{2})^2)$ and $\phi((\sqrt{2})^2) = (\phi(\sqrt{2}))^2 = (c + d\sqrt{5})^2$. Thus we have

$$
2 = c^2 + 5d^2 + 2cd\sqrt{5}
$$

which implies that $cd = 0$ and so either $c = 0$ or $d = 0$. If $c = 0$ then $5d^2 = 2$ which implies that $d = \sqrt{\frac{2}{5}}$ which is impossible since $d \in \mathbb{Q}$. Likewise, if $d = 0$ then we have $c^2 = 2$ which implies that $c = \sqrt{2}$ which is also impossible since $c \in \mathbb{Q}$. □
Note: A more delicate argument is as follows. Since the equation,

\[(\sqrt{2})^2 - 2 = 0\]

holds in \( \mathbb{Q}(\sqrt{2}) \) then if we apply a homomorphism \( \phi : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{5}) \) to this equation we get

\[\phi((\sqrt{2})^2 - 2) = \phi(0)\]
\[\phi((\sqrt{2})^2) - \phi(2) = \phi(0)\]
\[\phi(\sqrt{2})^2 - 2 = 0 \text{ since } \phi(2) = 2 \text{ and } \phi(0) = 0\]

But this says that \( \phi(\sqrt{2}) \), which is an element of \( \mathbb{Q}(\sqrt{5}) \), satisfies the equation \( x^2 - 2 = 0 \), but this says that \( \sqrt{2} \in \mathbb{Q}(\sqrt{5}) \) which is impossible.