with(linalg): with(plottools): Digits := 4:

# Here we choose a 4x4 matrix with complex entries.
A := array(1..4, 1..4, [[-30-30*I, 5+10*I, 7, -2], [-3-I, 2+5*I, 1+I, 0+I], [1-5*I, 1+3*I, 2+(-5)*I, 1], [2*I, 4, 2+7*I, 20+20*I]]):

f := charpoly(A, x); collect(f, x);

# Now let's numerically determine the eigenvalues of A, this uses certain polynomial techniques that are not covered in this class.
ev := fsolve(f, x, complex):
ev[1]; ev[2]; ev[3]; ev[4];

# Now let's determine the centers of the Gerschgorin disks.
# Here Re() means real part and Im() means imaginary part, e.g.
# Re(2+3I) = 2


c1 := [-30, -30]
c2 := [2, 5]
c3 := [2, -5]
c4 := [20, 20]

# Now compute the radii of the disks, recall this is the sum of the absolute values (norms) of the off diagonal entries in each row.
r1 := 20.18
r2 := 5.576
r3 := 9.261
r4 := 13.28

# Now create the disks so that we can plot them onscreen.
D1 := circle(c1, r1, color = red): D2 := circle(c2, r2, color = green): D3 := circle(c3, r3, color = blue): D4 := circle(c4, r4, color = orange):
D1 := circle(c1, r1, color=red):
D2 := circle(c2, r2, color=green):
D3 := circle(c3, r3, color=blue):
D4 := circle(c4, r4, color=black):
plots[display](D1, D2, D3, D4);

# Observe that the eigenvalues are pretty close to the centers of these four disks.

# Now, let’s see what happens if we try and shrink the radius of the disk in the upper right quadrant by

# by a factor of 10. We do this by conjugating A by R where R is
R := array(1..4, 1..4, [[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 10]]);

# The most important observation to make is that B has the same eigenvalues as A

# and so we shall determine the Gerschgorin disks (circles) for B but these will look a bit different.
# Actually, the centers won’t change (i.e. look at the diagonal of B vs. A) but the radii will change.

\[
B := \begin{bmatrix}
-30 - 30 I & 5 + 10 I & 7 & -20 \\
-3 - I & 2 + 5 I & 1 + I & 10 I \\
1 - 5 I & 1 + 3 I & 2 - 5 I & 10 \\
\frac{1}{5} & \frac{2}{5} & \frac{1}{5} + \frac{7}{10} I & 20 + 20 I
\end{bmatrix}
\]

\[
c_1 := [\text{Re}(B[1,1]), \text{Im}(B[1,1])];
c_2 := [\text{Re}(B[2,2]), \text{Im}(B[2,2])];
c_3 := [\text{Re}(B[3,3]), \text{Im}(B[3,3])];
c_4 := [\text{Re}(B[4,4]), \text{Im}(B[4,4])];
\]

\[
c_1 := [-30, -30] \\
c_2 := [2, 5] \\
c_3 := [2, -5] \\
c_4 := [20, 20]
\]

\[
r_1 := \text{evalf}(\text{abs}(B[1,2]) + \text{abs}(B[1,3]) + \text{abs}(B[1,4]));
r_2 := \text{evalf}(\text{abs}(B[2,1]) + \text{abs}(B[2,3]) + \text{abs}(B[2,4]));
r_3 := \text{evalf}(\text{abs}(B[3,1]) + \text{abs}(B[3,2]) + \text{abs}(B[3,4]));
r_4 := \text{evalf}(\text{abs}(B[4,1]) + \text{abs}(B[4,2]) + \text{abs}(B[4,3]));
\]

\[
r_1 := 38.18 \\
r_2 := 14.58 \\
r_3 := 18.26 \\
r_4 := 1.328
\]

\[
D_1 := \text{circle}(c_1, r_1, \text{color} = \text{red}); D_2 := \text{circle}(c_2, r_2, \text{color} = \text{green}); D_3 := \text{circle}(c_3, r_3, \text{color} = \text{blue}); D_4 := \text{circle}(c_4, r_4, \text{color} = \text{black});
\]

\[
\text{plots}([\text{display}](D_1, D_2, D_3, D_4));
\]
So we now see definitively that one of the eigenvalues will lie in the tiny disk in the upper right.

Note also that although the upper right disk shrunk, the others were enlarged a bit by this process.

Indeed, we could do this in reverse, and say make the lower left disk really large and thereby see that the eigenvalues reside within one large disk (that contains all the other disks).