MA 411 Homework 1

1. Write an equation for the line which goes through the points (3, 2) and (7, 0).

2. Write an equation for the plane which goes through the points (1, 0, 0), (0, 1, 0), and (0, 0, 1).

3. Find a vector perpendicular to the plane given in problem 2.

4. Write an equation for the plane which goes through the points (0, 0, 0), (1, 2, 3), and (−1, 0, 1).

5. Compute the kernel of the linear transformation $T$ from $V^3$ to $V^3$ given by
   \[Tx = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix} x.\]

6. Use Euclidean geometry to determine the area of the parallelogram with vertices at the points (0, 0), $(A, B)$, $(C, D)$, and $(A + C, B + D)$. Hint: Break up a large rectangle into smaller rectangle, right triangles, and the parallelogram.

7. Find the area of the parallelogram with vertices (0, 0), $(A, C)$, $(B, D)$, and $(A + B, C + D)$.

8. (a) Find the intersection point of the two lines $Ax + By = 3$ and $Cx + Dy = 5$.
   (b) Are there any values of the constants $A, B, C,$ and $D$ for which the lines don’t intersect?
      If so what are they?
   (c) Are there any values of the constants for which there is more than one intersection point?
      If so what are they?

9. What is the relationship between problems 7 and 8?

10. Without doing the computation, discuss how problems 6 through 9 generalize to 3 dimensions?

11. Consider the matrix $B = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$, and the associated linear mapping $T(x) = Bx$.
    
    (a) Compute $Bx$ and graph $x$ and $Bx$ on the same set of coordinates for each of the following values of $x$: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$.
    
    (b) Find the kernel of $T$.
    
    (c) Is $T$ one-to-one?
    
    (d) Find all $x$ so that $T(x) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.
    
    (e) Find the range of $T$.
    
    (f) Determine whether or not $T$ maps $V^2$ onto $V^2$. 

1
12. Consider the matrix \( D = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} \).

(a) Compute \( Dx \) and graph \( x \) and \( Dx \) on the same set of coordinates for each of the following values of \( x \): \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), \( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \), and \( \begin{bmatrix} -1 \\ -1 \end{bmatrix} \).

(b) Interpret the linear transformation \( T(x) = Dx \) geometrically.

13. Write down the definition of a neighborhood.

14. Write down the definition of an open set.

15. Prove that the set \( \mathcal{D} = \{(x_1, x_2) \in \mathbb{R}^2 \text{ such that } x_1^2 + x_2^2 < 1\} \) is open in \( \mathbb{R}^2 \). To do this you must show that for every \( x \) in \( \mathcal{D} \) there is a positive real number \( \delta = \delta(x) \) which has the following property: For each \( y \) in \( \mathbb{R}^2 \) whose distance to \( x \) is less than \( \delta \), \( y \) is in \( \mathcal{D} \).

16. Consider \( \delta = \delta(x_1, x_2) \) from the previous problem as a real valued function of two variables.

(a) Sketch the function by drawing its level sets.

(b) Where is the maximum of the function?

(c) * What is the most direct (i.e. steepest) path of ascent from the point \( \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} \) to the maximum?

17. Consider the surface \( z = \sin(x + y) \) formed by the graph of the function \( f(x, y) = \sin(x + y) \).

(a) Could an ant on this surface walk from the point \( (\pi, \pi) \) to \( (-5, 5 + 3\pi) \) without climbing? why or why not?

(b) Could an ant walk from the point \( (3\pi, 2\pi) \) to \( (2\pi, 4\pi) \) without climbing? why or why not?