
Above are five bifurcation diagrams A-E and below are five families of differential equations, each of which depends on a parameter \( A \). Match the number of the differential equation to the letter of the bifurcation diagram. If no bifurcation diagram corresponds to a given equation, write NONE next to the given number.

1. \( y' = Ay - y^3 \)  
2. \( y' = y^2 - A \)  
3. \( y' = Ay - y^2 \)  
4. \( y' = A - y^2 \)  
5. \( y' = A + y^2 \)

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2. [20 Points] Linear Systems. Each of the following questions may have multiple correct answers. Place each correct letter in the box.

2A. Which of the following are eigenvalues for the matrix

\[
\begin{pmatrix}
0 & -3 \\
2 & 5
\end{pmatrix}
\]

A. 0   B. 1   C. 2   D. 3   E. -2   F. None

Answer(s) =

2B. Which of the following are eigenvectors for the previous matrix

\[
\begin{pmatrix}
0 & -3 \\
2 & 5
\end{pmatrix}
\]

A. \( \begin{pmatrix} 3 \\ 2 \end{pmatrix} \)  B. \( \begin{pmatrix} -3 \\ 2 \end{pmatrix} \)  C. \( \begin{pmatrix} 2 \\ 3 \end{pmatrix} \)  D. \( \begin{pmatrix} -3 \\ 3 \end{pmatrix} \)  E. \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \)  F. \( \begin{pmatrix} 1 \\ -1 \end{pmatrix} \)  G. None

Answer =

2C. Now sketch the phase plane for the linear system

\[
Y' = \begin{pmatrix} 0 & -3 \\ 2 & 5 \end{pmatrix} Y
\]

(i.e., same matrix as in the previous two problems).

2D. List all values of \( A \) for which the linear system

\[
Y' = \begin{pmatrix} A & 1 \\ -1 & 0 \end{pmatrix} Y
\]

undergoes a bifurcation.
3. [12 Points] First order equations. Each of the following questions may have multiple correct solutions. Place the letter of each correct solution in the box.

3A. Which of the following functions are particular solutions of 

\[ y'' + 2y' + y = \sin(2t) \]

\begin{align*}
A. \ e^{-t} + te^{-t} & \quad \quad \quad \quad \quad \quad B. \ e^{-t} \\
C. \ -\frac{3}{25} \sin(2t) - \frac{4}{25} \cos(2t) & \quad \quad \quad \quad \quad \quad D. \ -\frac{3}{25} \cos(2t) - \frac{4}{25} \sin(2t) \\
E. \ 4e^{-t} - 6te^{-t} - \frac{3}{25} \sin(2t) - \frac{4}{25} \cos(2t) & \quad \quad \quad \quad \quad \quad F. \ 6e^{-t} - 4te^{-t} - \frac{3}{25} \cos(2t) - \frac{4}{25} \sin(2t)
\end{align*}

Answer =
3B. The solution of the initial value problem

\[ \frac{dy}{dt} + y = -2e^t, \ y(0) = 0 \]

is:

A. \(2e^t - 2e^{-t}\) \quad B. \(-2e^t + 2e^{-t}\) \quad C. \(e^{-t} - e^t\)

D. \(e^t - e^{-t}\) \quad E. \(e^t - te^t - e^{-t}\) \quad F. None of these

Answer =

4. [15 Points] Laplace transforms.

4A. Which of the following is the inverse Laplace transform of

\[ Y(s) = e^{-3s} \left( \frac{2}{s^2 - 1} \right) \]

A. \(u_3(t)\sin(t - 3)\) \quad B. \(u_3(t)(1/2)\sin(t - 3)\)

C. \(u_3(t)(e^t - e^{-t})\) \quad D. \(u_3(t)(e^{t-3} - e^{-(t-3)})\)

E. \(u_3(t)(e^{-t} + e^t)\) \quad F. \(u_3(t)(e^{t+3} - e^{t-3})\)

G. None of these

Answer =

4B. Evaluate the following integral:

\[ \int_0^5 (\delta_2(t) + 3u_4(t))dt \]

A. 0 \quad B. 1 \quad C. 2 \quad D. 3 \quad E. 4 \quad F. \infty \quad G. None of these

Answer =
4C. Compute the Laplace transform of
\[ y(t) = \begin{cases} 
  e^t & \text{if } t < 3 \\
  1 & \text{if } t \geq 3 
\end{cases} \]

\[ \begin{align*}
A. \quad & \frac{1}{s-1} - \frac{e^{-3s}}{s-3} + \frac{1}{s-3} \\
B. \quad & \frac{1}{s-1} - \frac{e^{-3s}}{s} + \frac{e^{-3s}}{s} \\
C. \quad & \frac{1}{s-1} - \frac{e^{-3s}}{s-1} \frac{1}{s} + \frac{e^{-s}}{s} \\
D. \quad & \frac{1}{s-1} + e^{-3s} \frac{e^3}{s-1} \frac{1}{s} \\
E. \quad & \frac{1}{s-1} - e^{-3s} \frac{e^3}{s-1} + \frac{e^{-3s}}{s} \\
F. None of these 
\end{align*} \]

Answer =

5. [10 Points] Nonlinear systems.
5A. The equilibrium points for the system
\begin{align*}
x' & = y - x \\
y' & = x - y^2
\end{align*}
are:
A. Sink and source  B. Sink and saddle  C. Both saddles  D. Saddle and source  E. Spiral sink and saddle  F. None of these

Answer =

5B. As time goes to infinity, the solution of the system
\begin{align*}
x' & = x^2 - 1 \\
y' & = 1 - y^2
\end{align*}
that satisfies the initial condition \( x(0) = y(0) = 0 \) tends to:

A. (1, 1)  B. (1, -1)  C. (-1, 1)  D. (-1, -1)  E. Infinity  F. None of these
6. [12 Points] Second order equations. Six second-order equations and four \( y(t) \)-graphs are given below. The equations are all of the form

\[
y'' + py' + qy = \cos(\omega t)
\]

for various values of the parameters \( p, q, \) and \( \omega \). For each \( y(t) \)-graph, determine the second-order equation for which \( y(t) \) is a solution.

(1) \( p = 5, q = 2, \omega = 2 \)  (2) \( p = 5, q = 1, \omega = 3 \)  (3) \( p = 1, q = 1, \omega = 3 \)

(4) \( p = 5, q = 3, \omega = 1 \)  (5) \( p = 1, q = 3, \omega = 2 \)  (6) \( p = 1, q = 3, \omega = 1 \)

\[
\begin{array}{|c|}
\hline
a = & b = & c = & d = \\
\hline
\end{array}
\]
7. [16 Points] Nonlinear systems

In this problem, you should show all of your work. This is not multiple choice. For the following system of differential equations, restrict attention to the first quadrant \((x, y \geq 0)\). For this system:

1. find and determine the types of all equilibria;
2. sketch the nullclines;
3. then sketch a representative collection of solutions in the phase plane.

\[
\frac{dx}{dt} = x(-x - y + 70) \\
\frac{dy}{dt} = y(-x^2 - y^2 + 2500)
\]

800. [1,000,000 Points] True/False. I am an engineer. ______________________