Your work will be collected on Wednesday, October 22, in class.

1. Prove: For all \( x \in \mathbb{R} \) there is exactly one integer \( n \in \mathbb{Z} \) such that \( n \leq x < n + 1 \).

2. Prove that every infinite bounded subset of a compact metric space has a limit point.

3. Let \( \{s_n\} \) be an arbitrary real sequence and let \( E \) be the set of all extended real numbers \( x \in \mathbb{R} \cup \{\infty, -\infty\} \) such that there is a subsequence \( \{s_{n_i}\}_{i=1}^{\infty} \) with \( \lim_{i \to \infty} s_{n_i} = x \). Write down a careful proof of each of the following two statements:
   a. \( E \) is non-empty.
   b. For \( s \in \mathbb{R} \):
      \[ \lim_{n \to \infty} s_n = s \iff \limsup_{n \to \infty} s_n = \liminf_{n \to \infty} s_n = s. \]

4. Let \( E \) be a subset of a metric space \((X, d)\). We define the boundary of \( E \) to be the set
   \[ \text{bd}(E) := E \cap E^c. \]
   Prove that:
   a. Prove: \( \text{int}(E) \), \( \text{int}(E^c) \), and \( \text{bd}(E) \) are pairwise disjoint and that their union is \( X \).
   b. Prove: \( \overline{E} = \text{int}(E) \cup \text{bd}(E) \).
   c. Prove: \( \text{bd}(E) = \phi \) if and only if \( E \) is both open and closed.
   d. Give an example of a metric space \( X \) and a proper non-empty subset \( E \subseteq X \) such that \( \text{bd}(E) \) is empty.

5. Suppose we are given an infinite triangular array of non-negative real numbers
   \[
   \begin{array}{cccccc}
   p_{11} & | & p_{21} & | & p_{22} & | \\
   p_{31} & | & p_{32} & | & p_{33} & | \\
   & | & \cdots & | & \cdots & | \\
   \end{array}
   \]
   in which the sum of the numbers in any row is 1 (i.e. \( p_{nm} \geq 0 \) and for each \( n \), \( p_{n1} + p_{n2} + \cdots + p_{nn} = 1 \)). Now let \( \{s_n\} \) be any sequence of real numbers and define a new sequence \( \{t_n\} \) by
   \[ t_n := p_{n1}s_1 + p_{n2}s_2 + \cdots p_{nn}s_n. \]
   a. Prove: for each \( n \), \( t_n \) lies somewhere between the maximum and the minimum of the numbers \( s_1, \ldots, s_n \).
   b. Now suppose also that \( \{s_n\} \) is a convergent sequence and that \( \lim_{n \to \infty} s_n = s \). Prove:
      \[ \lim_{n \to \infty} t_n = s \] if and only if for every positive integer \( m \), \( \lim_{n \to \infty} p_{nm} = 0 \).
c. Prove:
\[
\lim_{n \to \infty} \frac{1 + \frac{1}{2} + \cdots + \frac{1}{n}}{\log(n)} = 1.
\]

Hint: Use parts a and b with 
\[s_n = \frac{1}{n \cdot \log \frac{n+1}{n}}\]
and 
\[p_{nm} = \frac{\log \frac{n+1}{n}}{\log(n+1)}\].

An Extra Credit Problem.

6. (Extra Credit) Let \( p \) be a positive prime number and define the \( p \)-adic abolute value of any nonzero integer \( a \in \mathbb{Z} \) by

\[|a|_p = p^{-n}, \text{ where } p^n \text{ is the highest power of } p \text{ dividing } a.\]

We also define \( |0|_p = 0 \). For any two integers \( a, b \in \mathbb{Z} \) we then define the \( p \)-adic distance from \( a \) to \( b \) by

\[d_p(a, b) := |a - b|_p.\]

Note that two integers \( a \) and \( b \) are “close” in the \( p \)-adic metric if \( a \) and \( b \) are the same modulo a high power of \( p \).

a. Prove that \( d_p \) is a metric on \( \mathbb{Z} \).

Let \( X \) be the metric space \((\mathbb{Z}, d_p)\) (i.e. \( \mathbb{Z} \) with the \( p \)-adic metric).

b. Show that every series of the form \( \sum_{n=0}^{\infty} a_n p^n \) with \( a_n \in \mathbb{Z} \) is Cauchy.

c. Does the series \( (p-1) + (p-1)p + (p-1)p^2 + \cdots \) converge in \( X \)? If so, what is the limit? Does the series \( 1 + p + p^2 + \cdots \) converge in \( X \)?

d. Show that every Cauchy sequence is equivalent to exactly one series of the form

\[\sum_{n=0}^{\infty} a_n p^n, \text{ where each } a_n \in \mathbb{Z} \text{ satisfies } 0 \leq a_n < p.\]

e. Now let \( p = 5 \). Can you find a Cauchy sequence in \( X \) whose square converges to \(-1\)?