MA/BI 196 Final Exam

Name:

Instructions: The final exam is take home, and you are free to use your book, notes, or calculator. However, you are NOT to discuss the exam with anyone other than your instructor until after the final deadline; this includes fellow students in the class and those not in the class. If you have any questions, please email me.

A reasonable time frame for completing the exam is 3 hours, and I strongly suggest using only that amount of time. To receive full credit, you must show all work and fully explain your answer. Clearly designate your final answer for each problem.

The due date for the exam is Thursday, May 10th, at Noon. Ways you can turn in your exam include emailing me a scanned copy, leaving in my office MCS 230 (either if I am there or not), or leaving in my mailbox in the main Math Office (MCS 142). If you do not turn in your exam to me in person, I will confirm via email I have received it.

Good luck and have a great summer!

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Formulas:

Solution to second order discrete equation

The equation
\[ y_{t+1} = \alpha y_t + \beta y_{t-1}, \]
has the general solution
\[ y_t = a\lambda_1^t + b\lambda_2^t \]
where
\[ \lambda_{1,2} = \frac{1}{2} \left( \alpha \pm \sqrt{\alpha^2 + 4\beta} \right), \]
a, b are constants determined from \( y_0, y_1 \).

Converting a system to single second order equation

The system
\[ \begin{align*}
x_{t+1} &= ax_t + by_t \\
y_{t+1} &= cx_t + dy_t
\end{align*} \]
is equivalent to the second order equation
\[ x_{t+1} = (a + d)x_t + (bc - ad)x_{t-1} \]
1. (20 points) Compare the following continuous and discrete dynamical systems:

\[ x'(t) = 1 - \frac{x^2}{r}, \quad x_{t+1} = x_t + 1 - \frac{x_t^2}{r} \]

where \( r \) is a constant parameter.

(a) Find all equilibria and their stabilities; draw a bifurcation diagram as a function of \( r \) for each.
(b) For what values of \( r \) are the long-term dynamics of the two equations similar? dissimilar?
2. (20 points) Consider the following whale population model. If the number of whales falls below a minimum survival level, \( m \), then the species will become extinct. In addition, the population is limited by their environment, that is, if the whale population is above \( M \), then it will experience a decline because the environment cannot sustain that large of a population. If \( a_t \) represents the whale population (in hundreds) after \( t \) years,

\[
a_{t+1} = a_t + (M - a_t)(a_t - m).
\]

(a) Does the model seem plausible in terms of the description? Give a short explanation why or why not.

(b) Plot the updating function when \( M = 100, m = 1 \).

(c) Find all equilibria and determine their stability.

(d) The model has two serious shortcomings, what are they? Hint: What happens if \( a_0 < m \) or \( a_0 >> M \) (\( a_0 \) is a lot bigger than \( M \)).

(e) Bonus: How would you fix these two shortcomings?
3. (20 points) The following model is a refinement of the fishery model on the second midterm. The differential equation for the fish population, $F(t)$, where $t$ is measured in weeks, and $h$ is the harvesting rate, is

$$F'(t) = F \left(1 - \frac{F}{100}\right) - hF$$

This model incorporates a harvesting rate dependent on the fish population. This is plausible because when fewer fish are available, it is harder to find them so the weekly harvest drops.

(a) Find all equilibria and determine their stability when $h = 0.75$.

(b) Show that $F^* = 0$ undergoes a bifurcation when $h = 1$. What type is it? (Either give the name or describe in words).
4. (20 points) Fibonacci Rabbits revisited
   As before, a pair of immature rabbits is placed in a field. They mature after one month and reproduce, creating now \textit{two} new pairs of rabbits each subsequent month.

   (a) Write down a second order discrete equation for the number of rabbits in the field after \( t \) months; How does this compare to the original Fibonacci problem?

   (b) Solve the equation above.
5. (20 points) The following problem is called the “gambler’s ruin”. Suppose you are playing a game, where if you win, you gain $1, but if you lose, you lose $1. The probability of winning is $\alpha$, and thus the probability of losing is $1 - \alpha$. You go in with the following plan: you have $i$ to start, you always bet $1$ in each round and you stay until you either lose all your money or until you reach $100$. What is the probability of not being ruined (going broke)?

This problem is equivalent to the following second order discrete equation, where $P_i$ represents the probability of reaching $100$ (not being ruined) if you start with $i$; it satisfies the equation

$$P_{i+1} = \frac{1}{\alpha} P_i - \frac{1 - \alpha}{\alpha} P_{i-1},$$

with $P_0 = 0$, $P_{100} = 1$.

(a) Find the solution for $P_i$ when $\alpha = 0.25$; find $P_{90}, P_{95}$.
(b) Find the solution for $P_i$ when $\alpha = 0.75$; find $P_5, P_{10}$.
(c) With the above in mind, under what conditions would you advise someone to gamble in this way?
6. (20 points) The basic epidemic model consists of three classes, \( S \) (susceptible), \( I \) (infected), and \( R \) (recovered). Susceptible become infected at a rate proportional to the product of the number of infected and susceptible. Infected recover at a rate proportion to the number of infected. Once recovered, individuals are immune. The equations for \( S \) and \( I \) modeling this situation are

\[
\frac{dS}{dt} = -\alpha SI
\]

\[
\frac{dI}{dt} = \alpha SI - \mu I
\]

(a) Modify the above system of equations to account for the following addition. Both susceptible and infected individuals give birth at a rate \( b \) and all offspring are susceptible.

(b) Sketch the null-clines and find the equilibria if \( b = 1, \alpha = 2, \mu = 2 \).

(c) Does the infection persist or die out?