1. (12 points) Answer the following questions about 3D vector geometry.

(a) (3 pts) Find a vector which is normal to the plane $2x - 3z = 1$.

Solution: Copying the coefficients of $x$, $y$ and $z$ and pasting them into a vector gives $(2, 0, -3)$, which is a normal vector to this plane.

(b) (3 pts) Find the angle between the vectors $(1, 0, 1)$ and $(-1, 1, 0)$.

Solution: The angle $\theta$ satisfies

$$\cos \theta = \frac{(1, 0, 1) \cdot (-1, 1, 0)}{|(1, 0, 1)| |(-1, 1, 0)|} = \frac{-1}{\sqrt{2} \sqrt{2}} = -\frac{1}{2},$$

so that $\theta = \frac{2\pi}{3}$.

(c) (3 pts) If $\mathbf{v}$ and $\mathbf{w}$ are any vectors in space, then $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w})$ equals:

Solution: The cross product $\mathbf{v} \times \mathbf{w}$ is orthogonal to $\mathbf{v}$, so their dot product must equal 0.

(d) (3 pts) Write down the interpretation of $|\mathbf{v} \times \mathbf{w}|$ as an area.

Solution: $|\mathbf{v} \times \mathbf{w}|$ is the area of a parallelogram whose sides are $v$ and $w$. 
2. (6 points) Let \( \mathbf{F} = \langle f, g \rangle \) be a vector field with continuous partial derivatives on a simply-connected region \( R \) in the plane. Which of the following statements does not mean the same as the others?

(1) The divergence of \( \mathbf{F} \) is 0 everywhere on \( R \).
(2) \( \partial g/\partial x - \partial f/\partial y = 0 \) everywhere on \( R \).
(3) \( \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \) for every closed curve \( C \) in \( R \).
(4) \( \mathbf{F} = \nabla \phi \) for a function \( \phi \) defined on \( R \).

Solution: The answer is (1). (2) means that the curl of \( \mathbf{F} \) is 0, which means that \( \mathbf{F} \) must be conservative (4). If \( \mathbf{F} \) is conservative, then it has zero circulation around any closed curve (3). The statement (1) means that \( \mathbf{F} \) is source-free, which means that \( \mathbf{F} \) has zero flux through any closed curve, but that is something different.

3. (6 points) Suppose that \( x, y \) and \( z \) are positive numbers satisfying \( x + 2y + z = 12 \). Find the largest possible value of \( xyz \).

Solution: Let’s use Lagrange multipliers. We’re trying to maximize \( f(x, y, z) = xyz \) subject to the constraint \( g(x, y, z) = x + 2y + z = 12 \). So there’s going to be a scalar \( \lambda \) for which \( \nabla f = \lambda \nabla g \). This means
\[
\langle yz, xz, xy \rangle = \lambda \langle 1, 2, 1 \rangle,
\]
from which we get equations
\[
yz = \lambda \\
xz = 2\lambda \\
xy = \lambda
\]
Plugging \( \lambda = yz \) into the second equation gives \( xz = 2yz \). We can cancel the \( z \) (because it is positive) to get \( x = 2y \). Doing the same with the third equation gives \( x = z = 2y \). So, \( x = z = 2y \). Finally we have the original equation \( x + 2y + z = 12 \), which means that \( 2y + 2y + 2y = 12 \), or \( y = 2 \). Thus \( (x, y, z) = (4, 2, 4) \), and the largest value of \( xyz \) is \( 4 \times 2 \times 4 = 32 \).
4. (8 points) Let $\mathbf{F} = (xy, x + y)$. Let $C$ be the triangle in the plane with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$, oriented counterclockwise. Find $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

Solution: You can do three line integrals, but it’s easier to use Green’s theorem (the circulation form). This says that $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{curl} \, \mathbf{F} \, dA$, where $R$ is the region enclosed by $C$. This is the region bounded by $y = 0$, $x = 1$ and $y = x$. The curl of $\mathbf{F}$ is $1 - x$. We get

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{curl} \, \mathbf{F} \, dA$$
$$= \int_{x=0}^{1} \int_{y=0}^{x} (1 - x) \, dy \, dx$$
$$= \int_{x=0}^{1} \left. (1 - x)y \right|_{0}^{x}$$
$$= \int_{x=0}^{1} x - x^2 \, dx = \frac{1}{2} x^2 - \frac{1}{3} x^3 \bigg|_{0}^{1} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

5. (6 points) Find the Jacobian of the transformation $x = ve^u$, $y = ve^{-u}$.

Solution: The Jacobian is the determinant of the partial derivatives:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} ve^u & e^u \\ -ve^{-u} & e^{-u} \end{vmatrix} = v - (-v) = 2v.$$
6. (10 points) Let \( z = f(x, y) \) be a differentiable function. Let \( r \) and \( \theta \) be the usual variables from polar coordinates, so that \( x = r \cos \theta \) and \( y = r \sin \theta \). At the point \((r, \theta) = (1, \pi/4)\), find \( \partial z/\partial r \) and \( \partial z/\partial \theta \) in terms of \( \partial z/\partial x \) and \( \partial z/\partial y \).

Solution: The chain rule says that

\[
\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} = \cos \theta \frac{\partial z}{\partial x} + \sin \theta \frac{\partial z}{\partial y} = \sqrt{2} \frac{\partial z}{\partial x} + \sqrt{2} \frac{\partial z}{\partial y}.
\]

It’s similar with \( \frac{\partial z}{\partial \theta} \).

7. (6 points) Sketch the graph of \( z = \sqrt{x^2 + y^2} \).

Solution: This is a cone facing upward, with its vertex at the origin.
8. (10 points) Let $S$ be the filled-in square in the $xz$-plane with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(1, 0, 1)$ and $(0, 0, 1)$. Orient $S$ in the positive $y$-direction. Let $\mathbf{F}$ be the vector field $(-y, z, x)$. Find the flux of $\mathbf{F}$ through $S$.

Solution: The flux is $\int_S \mathbf{F} \cdot \mathbf{n} d\sigma$. Since $S$ is just a flat square in the $xz$-plane, the normal vector is just $\mathbf{n} = \mathbf{j} = (0, 1, 0)$. On $S$, the vector field is $\mathbf{F} = (0, z, x)$, because $y$ is always 0. We get
\[
\int_S \mathbf{F} \cdot \mathbf{n} d\sigma = \int_0^1 \int_0^1 \langle 0, z, x \rangle \cdot \langle 0, 1, 0 \rangle \; dz \; dx = \int_0^1 \int_0^1 z \; dz \; dx = \frac{1}{2}.
\]

9. (10 points) Find $\iiint_D z^2 dV$, where $D$ is the region in space defined by the inequalities $x^2 + y^2 \leq 4$, $0 \leq z \leq 1$.

Solution: $D$ is part of a cylinder, so let’s use cylindrical coordinates:
\[
\iiint_D z^2 \; dV = \int_0^{2\pi} \int_0^2 \int_0^1 z^2 \; dz \; dr \; d\theta = (2\pi)(2^2/2)(1/3) = 4\pi/3.
\]
10. (10 points) Let \( \phi(x, y, z) = x^3y + z \). Find \( \int_C \nabla \phi \cdot d\mathbf{r} \), where \( C \) is any curve starting at \((1, 0, 0)\) and ending at \((1, 1, 1)\).

By the fundamental theorem of line integrals,

\[
\int_C \nabla \phi \cdot d\mathbf{r} = \phi(1, 1, 1) - \phi(1, 0, 0) = 2 - 0 = 2.
\]