1. Decide if $(0, -4, -14, -40)$ is in the span of the vectors $(1, 1, 1, 1), (1, 2, 4, 8)$ and $(1, 3, 9, 27)$ in $\mathbb{R}^4$.

**Solution.** It is.

2. Let $P_3(\mathbb{R})$ be the vector space of polynomials with real coefficients of degree at most 3. Let $W \subset P_3(\mathbb{R})$ denote the subspace consisting of polynomials $f$ which satisfy the equations

\[
\begin{align*}
f(0) &= f'(0) \\
f(1) &= f'(1).
\end{align*}
\]

Find a basis for $W$.

**Solution.** Any polynomial in $P_3(\mathbb{R})$ is of the form $f(x) = ax + bx + cx^2 + dx^3$. The given equations imply that

\[
\begin{align*}
a &= b \\
a + b + c + d &= b + 2c + 3d,
\end{align*}
\]

a system of two linear equations in four unknowns. Eliminating $b$ from the second equation gives $a = c + 2d$. Thus we may take $c$ and $d$ to be our independent variables (parameters), and $a$ and $b$ depend on these. A basis for $W$ is given by setting $(c, d)$ as $(1, 0)$ and $(0, 1)$, respectively, giving the basis \{1 + x + x^2, 2 + 2x + x^3\}.

3. Let $V$ be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Let $T: V \rightarrow V$ be the function defined by $(Tf)(x) = f(x^2)$. Thus for instance $T$ takes the function $\sin(x)$ to $\sin(x^2)$. Is $T$ a linear transformation? Briefly justify your answer.

**Solution.** It is. The appearance of the $x^2$ term is just a red herring. To check that $T$ is linear, we must show that for $f, g \in V$ and $c \in \mathbb{R}$, we have $T(f + g) = T(f) + T(g)$ and $T(cf) = cT(f)$. To check that $T(f + g) = T(f) + T(g)$, we can show that $T(f + g)(x) = (T(f) + T(g))(x)$ for all $x \in \mathbb{R}$. We have

\[
T(f + g)(x) = (f + g)(x^2) = f(x^2) + g(x^2) = T(f)(x) + T(g)(x) = (T(f) + T(g))(x).
\]
A similar argument works for $T(cf) = cT(f)$. On the other hand, if $U: V \to V$ is the function defined by $(Uf)(x) = f(x)^2$, then $U$ is not a linear transformation.

4. Let $T: \mathbb{R}^3 \to \mathbb{R}$ be a linear transformation. Show that there exist scalars $a$, $b$ and $c$ such that $T(x, y, z) = ax + by + cz$ for all $(x, y, z) \in \mathbb{R}^3$.

**Solution.** Let

\[
\begin{align*}
a &= T(1, 0, 0) \\
b &= T(0, 1, 0) \\
c &= T(0, 0, 1)
\end{align*}
\]

Then for all $(x, y, z) \in \mathbb{R}^3$, we have

\[
T(x, y, z) = T(x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)) = xT(1, 0, 0) + yT(0, 1, 0) + zT(0, 0, 1) = ax + by + cz
\]

5. Let $T: V \to W$ be a linear transformation between two vector spaces. Show that if $v_1, \ldots, v_n \in V$ are vectors and $\{T(v_1), \ldots, T(v_n)\}$ is linearly independent, then so is $\{v_1, \ldots, v_n\}$.

**Solution.** Suppose that there are scalars $a_1, \ldots, a_n$ with

\[
a_1v_1 + \cdots + a_nv_n = 0.
\]

Applying $T$ (and using the fact that $T$ is linear) gives

\[
a_1T(v_1) + \cdots + a_nT(v_n) = 0.
\]

Since $\{T(v_1), \ldots, T(v_n)\}$ is linearly independent, all of the $a_i$ are zero. This shows that $\{v_1, \ldots, v_n\}$ is linearly independent.

6. Let $T: V \to W$ be a linear transformation between two vector spaces, and let $U: W \to X$ be another. Recall that $UT: V \to X$ is the linear transformation defined by $UT(v) = U(T(v))$ for all $v \in V$. Show that if $T$ and $U$ are both one-to-one, then so is $UT$.

**Solution.** A linear transformation is one-to-one if and only if its nullspace is 0. Let $v$ be a vector in the nullspace of $UT$. Then $UT(v) = 0$, so that $U(T(v)) = 0$. This means that $T(v)$ lies in the nullspace of $U$, but since $U$ is one-to-one, $T(v) = 0$. Thus $v$ is in the nullspace of $T$, but since $T$ is one-to-one, $v = 0$. We have shown that the only vector in the nullspace of $UT$ is the zero vector, so that $UT$ is one-to-one.