1 Admissible representations of a locally profinite group

1.1 Admissibility

Let $G$ be locally profinite. Recall that a representation $\pi : G \to \text{GL}(V)$ is admissible if it is smooth and has the property that $V^K$ is finite-dimensional for all open compact $K \subset G$.

If $\sigma : K \to \text{GL}(W)$ is a representation of $K$, let $V[\sigma]$ be the $\sigma$-isotypic subspace of $V$. This is the sum of the images of all $K$-equivariant maps $W \to V$.

**Proposition 1.1.** A representation $\pi : G \to \text{GL}(V)$ is admissible if and only if the following two conditions hold:

1. For all irreducible representations $\sigma$ of $K$, $V[\sigma]$ is finite-dimensional.
2. $V = \bigoplus_\sigma V[\sigma]$, where $\sigma$ runs over all smooth irreducible representations of $K$.

**Proof.** Left as exercise. \hfill $\square$

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\textsuperscript{1}Much of this lecture is adapted from David Rohrlich’s manuscript on automorphic representations.
1.2 Schur’s lemma

As a consequence we have Schur’s lemma for irreducible admissible representations \( \pi: G \to \text{GL}(V) \). This states that \( \text{Hom}_G(V,V) = \mathbb{C} \). Proof: \( f: V \to V \) commutes with the action of \( G \), we would like to show that \( f \) has an eigenvalue \( \lambda \). (Note that the existence of an eigenvector for \( f \) is not at all guaranteed, because \( V \) may be infinite-dimensional.) Then the \( \lambda \)-eigenspace of \( f \) is a nonzero \( G \)-stable subspace of \( V \), hence it is all of \( V \), hence \( f \) acts on \( V \) as \( \lambda \). To show that \( f \) has an eigenvalue, let \( K \) be an open compact subgroup. By the proposition, there is a smooth representation \( \sigma \) of \( K \) with \( V[\sigma] \) finite-dimensional and nonzero. Now observe that \( f \) preserves \( V[\sigma] \). Since \( V[\sigma] \) is finite-dimensional, it has an eigenvalue.

1.3 Admissible unitary representations

Unitary representations \( V \) have the nice property that for \( W \subset V \) a \( G \)-invariant subspace, the complement \( W^\perp \) is also \( G \)-invariant. This suggests that \( V \) should be a direct sum of irreducible \( G \)-invariant subspaces. But this may fail to be true: \( V \) might not even be equal to the direct sum of \( W \) and \( W^\perp \)! It turns out that we are saved by the property of admissibility.

**Theorem 1.2.** Let \( G \) be locally profinite, and let \( \pi: G \to \text{GL}(V) \) be an admissible unitary representation. Then \( V \) is the orthogonal direct sum of irreducible representations.

This will follow from the following two lemmas:

**Lemma 1.3.** If \( W \subset V \) is \( G \)-invariant, then \( V = W \oplus W^\perp \).

**Lemma 1.4.** If \( V \neq 0 \) then \( V \) contains an irreducible \( G \)-invariant subspace.

Given the lemmas, the proof of the theorem proceeds as follows. Let \( \mathcal{S} \) be the set whose members are families of mutually orthogonal irreducible subspaces of \( V \). Put a partial order on \( \mathcal{S} \) under inclusion. Then every chain in \( \mathcal{S} \) has an upper bound, namely the union. Therefore by Zorn’s lemma, \( \mathcal{S} \) has a maximal element \( S \). Put

\[
U = \bigoplus_{W \in S} W.
\]
We claim \( U = V \). Otherwise, the first lemma gives \( V = U \oplus U^\perp \), and the second lemma shows that \( U^\perp \) contains an irreducible representation \( W \). Then \( S \cup \{ W \} \) is strictly greater than \( S \), contradiction.

Now, the proof of Lemma 1: Since \( V \) is admissible, it enough to show that

\[
V[\sigma] = W[\sigma] \oplus W[\sigma]^\perp
\]

for every smooth irreducible representation of \( K \). Since \( V[\sigma] \) is finite-dimensional, we have

\[
V[\sigma] = W[\sigma] \oplus (W[\sigma]^\perp \cap V[\sigma])
\]

We claim that \( W[\sigma]^\perp = W[\sigma]^\perp \cap V[\sigma] \). Certainly \( W[\sigma] \subset V[\sigma] \) and \( W[\sigma]^\perp \subset W[\sigma]^\perp \), so that we have one inclusion, \( W[\sigma] \subset W[\sigma]^\perp \cap V[\sigma] \).

On the other hand, \( V[\sigma] \) is orthogonal to \( W[\tau] \) for all \( \tau \neq \sigma \), and \( W[\sigma]^\perp \) is orthogonal to \( V[\sigma] \), so that the intersection \( V[\sigma] \cap W[\sigma]^\perp \) is orthogonal to

\[
\left( \bigoplus_{\tau \neq \sigma} W[\tau] \right) \bigoplus W[\sigma] = W,
\]

so that \( V[\sigma] \cap W[\sigma]^\perp \subset W^\perp \), which implies that \( V[\sigma] \cap W[\sigma]^\perp \) (since it is \( \sigma \)-isotypic) is contained in \( W[\sigma]^\perp \). This is the other required inclusion.

Proof of Lemma 2: Left as exercise.

## 2 Representations of restricted direct product groups

Let \( G_v \) be a collection of locally profinite groups. For almost all \( v \), let \( K_v \subset G_v \) be a compact open subgroup. Let \( S_0 \) be the finite set of \( v \) containing those for which \( K_v \) is not defined. We need to make the following important assumption: the Hecke algebra \( H(G_v, K_v) \) is commutative for all \( v \not\in S \). This mimics the situation of \( G_v = \text{GL}_2(K_v), K_v = \text{GL}_2(O_v) \).

For all \( v \), suppose \( \pi_v : G_v \to \text{GL}(V_v) \) be a smooth admissible representation. Assume that for almost all \( v \not\in S_0 \), \( \pi_v \) is “spherical”, in the sense that \( V_v^{K_v} \neq 0 \). Assume that \( S \) contains all those \( v \) for which \( \pi_v \) is not spherical. For \( v \not\in S_0 \), \( \dim V_v^{K_v} = 1 \) is 1-dimensional (because the Hecke algebra is commutative); let \( \xi_v \in V_v \) be a nonzero spherical vector.
Now define the restricted tensor product $\pi = \bigotimes_v' \pi_v$ as follows. For every finite $S \supset S_0$, let
\[ \pi_S = \bigotimes_{v \in S} \pi_v; \]
this is a representation of $\prod_{v \in S} G_v$ on a vector space $V_S$. In fact it is a representation of
\[ G_S = \prod_{v \in S} G_v \times \prod_{v \notin S} K_v \]
because the $K_v$ act trivially on $V_S$. For $S \subset S'$, define a map $V_S \rightarrow V_{S'}$ by tensoring a vector $x \in V_S$ with $\otimes_{v \in S' \setminus S} \xi_v$. This map is compatible with the actions of $G_S$ and $G_{S'}$ and the inclusion $G_S \hookrightarrow G_{S'}$, because the $K_v$ act trivially on the $\xi_v$. Then set
\[ \bigotimes_v' \pi_v = \lim_{S \rightarrow S'} \pi_S; \]
this is a representation of
\[ G = \lim_{S \rightarrow S'} G_S. \]
Thus $\pi$ is the vector space spanned by symbols $\otimes_v x_v$, with $x_v = \xi_v$ for almost all $v \notin S_0$, modulo the usual relations regarding addition and scalar multiplication. Note that it is smooth and admissible (follows easily from the corresponding properties of the $\pi_v$). Also, if each $\pi_v$ is irreducible, then so is $\pi$.

Should each $\pi_v$ be a unitary representation with norm $\|x\|_v$, one can give $\pi$ the structure of a unitary representation. For this, one assumes that each $\xi_v$, $v \notin S$, is a unit vector with respect to the inner product on $V_v$. Then the norm of $\otimes_v x_v$ is defined to be $\prod_v \|x_v\|_v$.

A smooth representation $\pi$ of $G$ is decomposable if it isomorphic to some $\bigotimes_v' \pi_v$.

2.1 The factorizability theorem

**Theorem 2.1.** Suppose $\pi$ is a unitary admissible irreducible representation of $G = \prod_v' G_v$ on a vector space $V$. Then $\pi$ is decomposable, and its irreducible factors are uniquely defined.
Proof. The first observation is that for each $v$, $\pi|_{G_v}$ is the sum of subrepresentations which are all isomorphic to one particular representation, call it $\pi_v: G_v \to \text{GL}(V_v)$. Proof: $\pi|_{G_v}$ is again an admissible unitary representation, so it contains an irreducible $G_v$-invariant subspace $V_v$. Let $\pi_v$ be the representation of $G_v$ on $V_v$. Then (key observation coming up) $V[\pi_v]$ is preserved by the action of $G$. Therefore $V = V[\pi_v]$. By the previous theorem, $\pi|_{G_v}$ is the direct sum of representations all isomorphic to $\pi_v$.

We claim that $\pi_v$ is spherical for almost every $v$. Since $\pi$ is smooth, an arbitrary nonzero vector in $V$ is fixed by a compact open $K \subset G$. By definition of the topology on $G$, $K$ contains $K_v$ for almost all $v$. Since $\pi|_{K_v}$ is the direct sum of representations all isomorphic to $\pi_v$, $\pi_v$ has a $K_v$-fixed vector for almost all $v$. Enlarge $S_0$ to include all the $v$ for which $\pi_v$ is not spherical. Choose spherical unit vectors $\xi_v \in V_v$ for all $v \notin S_0$.

For each $v$ we may define $V^v$ to be the space of $G_v$-equivariant linear maps $V_v \to V$ which are smooth with respect to the evident action $\pi^v$ of $G^v = \prod'_{w \neq v} G_w$. Since $\pi_v$ embeds into $\pi$ by its very definition, there do exist nonzero $G_v$-equivariant linear maps $f: V_v \to V$. Furthermore these are automatically smooth. Indeed, let $x \in V_v$ is a nonzero vector, let $K_v \subset G_v$ fix $f(x)$. Then $K^v$ fixes $f(\pi_v(g)x)$ for all $g \in G_v$ (since the actions of $G_v$ and $G^v$ commute). Since $\pi_v$ is irreducible, the $\pi_v(g)x$ must span $V_v$. Thus $K^v$ fixes $f(y)$ for all $y \in V_v$, which is to say that $K^v$ fixes $f$. Therefore $\pi^v$ is nonzero.

We have a nonzero $G$-equivariant map $V_v \otimes V^v \to V$ given by “evaluation”: $x \otimes f \mapsto f(x)$, and since both sides are irreducible, we have an isomorphism $V \cong V_v \otimes V^v$.

This can be generalized: if $S$ is a finite set of indices, let $\pi_S$ be the representation of $\prod_{v \in S} G_v$ on $V_S = \bigotimes_{v \in S} V_v$; we have a factorization $V \cong V_S \otimes V^S$.

For $S$ containing $S_0$, we claim that $V^S$ has a unique spherical vector up to scaling. Existence is because $V$ itself has an $S$-spherical vector, hence so does $V^S$. Uniqueness is because $\mathcal{H}(\prod'_{v} G_v, \prod_v K_v)$ is commutative (this follows from the local statement). Let $\xi^S$ be a nonzero spherical vector in $V^S$.

Let $\pi_S$ be the representation of $G_S$ on $V_S$, defined by having $K_v$ act trivially for $v \notin S$. Whenever $S \subset S'$, we have a $G_S$-equivariant map $V_S \to$
$V_{\pi'}$ given by tensoring with $\otimes_{v \in S} \xi_v'$. We have $\otimes' \pi_v = \lim_{\rightarrow} \pi_S$ by definition. We must produce a nonzero $G$-equivariant map $\otimes' \pi_v \to \pi$. This is done by producing compatible maps $\pi_S \to \pi$ for each $S \supset S_0$ by means of the embedding

$$V_S \subset V_S \otimes V^S \cong V$$

given by tensoring with $\xi_S$. (One must perhaps rescale the $\xi_S$ to ensure compatibility.)

3 Representations of $GL_2(A_{\text{fin}}^\infty)$ arising from cusp forms

Let $k \geq 1$. Recall we have a space $S_k$ of adelic modular forms. This was the smooth induction from $GL_2^+(\mathbb{Q})$ up to $GL_2(A^\text{fin}_\mathbb{Q})$ of the space of holomorphic functions on $\mathcal{H}$ which vanish at the cusps, under the action $f \mapsto f|_{\gamma^{-1},k}$.

**Proposition 3.1.** $S_k$ is a unitary admissible representation of $GL_2(A_{\text{fin}}^\infty)$.

**Proof.** (Sketch) First, admissibility. Let $K \subset GL_2(A^\text{fin}_\mathbb{Q})$ be a compact open subgroup. We want to show that $S^K_k$ is finite-dimensional. First observe that the double coset space

$$GL_2^+(\mathbb{Q}) \backslash GL_2(A^\text{fin}_\mathbb{Q})/K$$

is finite. (By conjugating $K$ one may assume that it is a finite-index subgroup of $GL_2(\mathbb{Z})$.) Then use strong approximation: $GL_2(A^\text{fin}_\mathbb{Q}) = GL_2^+(\mathbb{Q}) GL_2(\mathbb{Z})$.)

Let $C$ be a set of coset representatives for this double coset space. For each $c \in C$, let $\Gamma_c = GL_2^+(\mathbb{Q}) \cap cKc^{-1}$. Then there is a map

$$S^K_k \to \bigoplus_{c \in C} S_k(\Gamma_c)$$

$$\phi \mapsto (\phi(c))_{c \in C},$$

which is in fact an isomorphism (Exercise). Therefore $S^K_k$ is finite-dimensional and $S_k$ is admissible. (In fact $M_k$ is admissible as well, by the same argument.)

To give $S_k$ the structure of an inner-product space, it will be useful to interpret elements of $S_k$ as functions not on $GL_2(A^\text{fin}_\mathbb{Q})$ but on all of $GL_2(A_{\mathbb{Q}})$. First note that

$$GL_2(A^\text{fin}_\mathbb{Q}) = GL_2(\mathbb{Q})(GL_2(A^\text{fin}_\mathbb{Q}) \times GL_2^+(\mathbb{R})), $$

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where $GL_2(Q)$ is embedded diagonally in $GL_2(A_Q)$.

Given $\phi \in S_k$, define a function $\Phi$ on elements

$$g = \gamma(g_{\text{fin}}, g_{\infty}) \in GL_2(Q)(GL_2(A_Q^{\text{fin}}) \times GL_2^+(R))$$

by

$$\Phi(g) = (\phi(g_{\text{fin}})|_{g_{\infty,k}})(i)$$

(Do check this is well-defined!) Then $\Phi$ is invariant under left multiplication by $GL_2(Q)$. It is also invariant under right multiplication by an open subgroup $K \subset GL_2(\hat{Z})$ (depending on $\Phi$). Finally, $\Phi$ is invariant under multiplication by the group $Z(R)^+$ of diagonal matrices

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in GL_2(R)$$

with $a > 0$, whereas

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

transforms $\Phi$ into $(-1)^k \Phi$. Let $\phi_1, \phi_2 \in S_k$ have images $\Phi_1, \Phi_2$ under this correspondence; we define

$$\langle \phi_1, \phi_2 \rangle = \int_{GL_2(Q)Z(R) \backslash GL_2(A_Q)} \Phi_1(g)\overline{\Phi_2(g)}d\mu(g)$$

Here $d\mu$ is a Haar measure on the locally compact group $Z(R) \backslash GL_2(A_Q)$; the integral is understood to mean the integral over a fundamental domain for $GL_2(Q)Z(R) \backslash GL_2(A_Q)$ inside of $Z(R) \backslash GL_2(A_Q)$. The integral is well-defined because the $\Phi_i$ are left $GL_2(Q)$-invariant, and the integrand is $Z(R)$-invariant. It is convergent because the $\phi_i$ were cusp forms, although we do not check the details here. Finally, $\langle \phi_1, \phi_2 \rangle$ is $GL_2(A_Q^{\text{fin}})$-invariant because $d\mu$ is a Haar measure. If the $\phi_i$ arose from cusp forms $f_i$ lying in a common space $S_k(\Gamma)$, then $\langle \phi_1, \phi_2 \rangle$ equals the Petersson inner product $\langle f_1, f_2 \rangle$ up to a scalar which only depends on $\Gamma$ (and on $d\mu$).

As a corollary, we find that $S_k$ decomposes as a direct sum of irreducible representations, each of which is an admissible representation $\pi$ of $GL_2(A_Q^{\text{fin}})$. By the factorizability theorem, each $\pi$ is the restricted direct product of representations $\pi_p$ of $GL_2(Q_p)$, almost all of which are spherical. By Casselmann’s theorem on the new vector, there exists for each $p$ an integer $c_p$ (almost always 0) and a unit vector $\phi_p$ in the space of $\pi_p$ for which

$$\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} \phi_p = \chi_p(a) \phi_p.$$
for all \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}_p) \) with \( c \equiv 0 \pmod{p^{\ell_0}} \). Here \( \chi_p \) is the central character of \( \pi_p \).

Note that \( \phi_p \) is a spherical vector for almost all \( p \), so that \( \phi = \otimes_p \phi_p \) is a well-defined unit vector in the space of \( \pi \). Let \( N = \prod_p p^{\ell_0} \). Then \( \phi \) is invariant under \( K_1(N) \). Since \( K_1(N) \) surjects via the determinant map onto \( \hat{\mathbb{Z}}^\times \), we have \( S_k^{K_1(N)} \cong S_k(\Gamma_1(N)) \), so that \( \phi = \phi_f \) for a classical cusp form \( f \in S_k(\Gamma_1(N)) \). Since the center of \( \text{GL}_2(\mathbb{A}_Q) \) acts on \( \phi \) through \( \chi \), we have \( f \in S_k(N, \chi) \).