We now initiate the study of automorphic forms on $\text{GL}_2$ as spaces of functions on the quotient $\text{GL}_2(\mathbb{Q})\backslash \text{GL}_2(\mathbb{A}_\mathbb{Q})$. It is this formulation that will allow us to generalize the notion of automorphic form to global fields other than $\mathbb{Q}$ and groups other than $\text{GL}_2$. The presence of archimedean places poses a significant hurdle, though. It will become necessary to devise a notion of admissibility for representations of $\text{GL}_2(\mathbb{R})$ parallel to the notion of admissibility for representations of locally profinite groups $G$ such as $\text{GL}_2(\mathbb{Q}_p)$. One very nice feature of such representations is that their restriction to a maximal compact subgroup $K$ decomposes as a direct sum of irreducible representations of $K$, each of which appears only finitely many times. One can then classify representations of $G$ according to which irreducible representations of $K$ they contain.

One problem with extending this definition to representations of (say) $\text{SL}_2(\mathbb{R})$ is that such representations, when infinite-dimensional, tend to be really huge (at least uncountable), whereas there are only countably many irreducible representations of the maximal compact subgroup $K = \text{SO}_2(\mathbb{R})$. (Note that $\text{SL}_2(\mathbb{Q}_p)/\text{SL}_2(\mathbb{Z}_p)$ is countable, while $\text{SL}_2(\mathbb{R})/\text{SO}_2(\mathbb{R})$ is the upper half-plane.) The workaround, developed by Harish-Chandra, is the notion of a $(\mathfrak{g}, K)$-module for a Lie group $G$. It isn’t actually a representation of $G$ at all!

Here is our motivation: suppose $\pi: G \to \text{GL}(V)$ is a continuous representation of $G$ on a Hilbert space $V$, but not necessarily a unitary one. Recall that $\text{Lie} G$ is the tangent space to the identity of $G$, and that each $X \in \mathfrak{g} = \text{Lie} G$ defines a one-parameter subgroup $t \mapsto \exp(tX)$ of $G$. A vector $v \in V$ is $C^1$ if, for all $X \in \mathfrak{g} = \text{Lie} G$, the derivative

$$\pi(X)v := \frac{d}{dt} \big|_{t=0} \pi(\exp(tX))v = \lim_{t \to 0} \frac{\pi(\exp(tX))v - v}{t}$$
is defined. The vector \( v \in V \) is \( C^\infty \), or smooth, if \( \pi(X_1) \cdots \pi(X_n)v \) is defined for every sequence \( X_1, \ldots, X_n \) of elements of \( \mathfrak{g} \). Let \( V^\infty \) be the subspace of smooth vectors. It is a representation of \( G \). It will be unlikely that \( V = V^\infty \), but for \( G = \text{GL}_n(\mathbb{R}) \) it is the case that \( V \) is dense in \( V^\infty \) (see Bump, p. 190).

There is a representation \( \pi : \mathfrak{g} \to \text{End} V^\infty \), called the *infinitesimal action*. This places us in the realm of representations of Lie algebras, but there is still a problem in that \( V^\infty \) is very large, yet generally not a Hilbert space.

Let \( K \subset G \) be a maximal compact subgroup of \( G \). By the Peter-Weyl theorem, \( V \) decomposes as a Hilbert direct sum of irreducible unitary representations of \( K \). We say \( V \) is *admissible* if each isomorphism class of irreducible representation of \( K \) appears only finitely often in such a decomposition.

Let \( \pi : G \to \text{GL}(V) \) be an admissible representation; assume that the restriction of \( \pi \) to \( K \) is unitary. (This can always be assumed, by averaging the inner product on \( V \) over \( K \).) We have

\[
V = \bigoplus V[\sigma],
\]

where \( \sigma \) runs over the set of isomorphism classes of unitary irreducible representations of \( K \). Then each \( V[\sigma] \) is finite-dimensional. Let \( V^\text{fin} \) be the algebraic direct sum

\[
V^\text{fin} = \bigoplus V[\sigma].
\]

Vectors lying in \( V^\text{fin} \) are called *\( K \)-finite*. A vector \( v \in V \) belongs to \( V^\text{fin} \) if and only if the space spanned by \( \pi(k)v, k \in K \), is finite-dimensional. \( V^\text{fin} \) has the following virtues (see Bump, p. 197):

1. \( V^\text{fin} \) is dense in \( V \),
2. \( V^\text{fin} \subset V^\infty \),
3. \( V^\text{fin} \) is invariant under the action of \( \pi(\mathfrak{g}) \),

but it also has the vice of not being \( G \)-invariant. Nevertheless, \( V^\text{fin} \) mels together two structures which are both quite algebraic (read: tractable) in nature, namely that it has an action of \( \mathfrak{g} \), and also an action of \( K \) with respect to which it is admissible.

A \((\mathfrak{g}, K)\)-module is a vector space \( V \) together with representations \( \pi : \mathfrak{g} \to \text{End}(V) \) and \( \pi : K \to \text{GL}(V) \) such that:

1. \( V \) is the (algebraic) direct sum of finite-dimensional irreducible representations of \( K \),
2. The infinitesimal action of $K$ on $G$ agrees with the restriction of $\pi : \mathfrak{g} \to \text{End}(V)$ to $\mathfrak{k}$.

3. For $X \in \mathfrak{g}, k \in K$, we have $\pi(k)\pi(X)\pi(k^{-1}) = \pi((\text{Ad} k)X)$ as operators on $V$.

Furthermore, $V$ is admissible if each irreducible representation of $K$ appears only finitely many times in $V$.

1 Classification of admissible $(\mathfrak{g}, K)$-modules for $\text{GL}_2(\mathbb{R})$

Let $G^+ = \text{GL}_2^+(\mathbb{R})$, $K = \text{SO}(2)$. Then $\mathfrak{g}$ is spanned by $r = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $l = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$, and $z = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $z$ lies in the center of $\mathfrak{g}$, whereas

\[ [h, r] = 2r \]
\[ [h, l] = -2l \]
\[ [r, l] = h. \]

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and let

\[ -4\Delta = h^2 + 2rl + 2lr \in U(\mathfrak{g}). \]

Then $\Delta$ lies in the center of $U(\mathfrak{g})$. Indeed this $\Delta$ is the Casimir operator (normalized to agree with convention).

Let $(\pi, V)$ be an admissible $(\mathfrak{g}, K)$-module. Since $K \cong \mathbb{R}/2\pi\mathbb{Z}$, $V$ is the direct sum of finite-dimensional spaces $V[k]$ on which \[ \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \] acts as the scalar $e^{ikt}$.

Notice that if $W = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$, and $x \in V[k]$, then

\[ \pi(W).x = \frac{d}{dt}\pi(e^{it}W)x|_{t=0} = \frac{d}{dt}\pi \left( \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \right)x|_{t=0} = \frac{d}{dt}e^{ikt}x|_{t=0} = ikx. \]
Thus if 
\[ H = -iW = \begin{pmatrix} -i \\ i \end{pmatrix} \in \mathfrak{g}_C, \]
then \( \pi(W) \) acts on \( V[k] \) as the scalar \( k \).

The element \( H \) has eigenvalues 1, \(-1\), just like \( h \). The two are conjugate: \( H = C^{-1}hC \), where \( C = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \). After conjugating by \( C \), one arrives at a new basis for \( \mathfrak{g}_C \):

\[
R = \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \\
L = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \\
H = \begin{pmatrix} i \\ -i \end{pmatrix} \\
Z = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

(Some of our notation differs from Bump’s, because Bump parametrizes \( SO(2) \) by the rotation matrix \( \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \).) This basis has the same commutation relations as \{r, l, h\}.

Simply write \( Xx \) for \( \pi(X)x \). If \( x \in V[k] \) we have

\[
HRx = (HR - RH)x + RH = [H, R]x + RH = 2Rx + kRx = (k + 2)Rx,
\]
so that \( RV[k] \subset V[k + 2] \). Similarly \( LV[k] \subset V[k - 2] \).

Now assume that \( V \) is irreducible. Then \( \Delta \) must act on \( V \) as a constant, say \( \lambda \). Let \( x \in V[k] \) be nonzero. Let \( U \) be the span of \( x, R^n x, L^n x \) \((n > 0)\). It is easy to show that \( U \) is invariant under both \( \mathfrak{g} \) and \( K = SO(2) \). (Invariance under \( K \) is self-evident, since \( K \) acts as a scalar on each \( V[\ell] \). Invariance under \( Z \) and \( \Delta \) follows from Schur’s lemma, since these guys act as scalars. Say \( \Delta \) acts as \( \lambda \) on \( V \). We have

\[
-4\lambda w = (\ell^2 + 2\ell)w + 4LRw,
\]
implying that if \( w \in CR^n x \), then \( LRw \in CRw \), etc.) Since \( V \) is irreducible, this sum must be all of \( V \). Therefore each \( V[n] \) is at most one-dimensional.
(and is zero unless \( n \equiv k \pmod{2} \)). Some more tinkering shows that if \( x \in V[k] \) is nonzero, and \( Rx = 0 \), then \( \lambda = -\frac{k}{2}(1 + \frac{k}{2}) \). Likewise if \( x \in V[k] \) is nonzero, and \( Lx = 0 \), then \( \lambda = \frac{k}{2}(1 - \frac{k}{2}) \). Indeed, if \( Rx = 0 \), then

\[
-4\lambda x = \Delta x = (H^2 + 2H + 4LR)x = (k^2 + 2k)x,
\]

etc.

Already this shows that if \( V \) is an irreducible \((\mathfrak{g}, K)\)-module, then there are four possibilities:

1. One is that \( V = \bigoplus_{k \equiv \varepsilon \pmod{2}} V[k] \), with each \( V[k] \neq 0 \) one-dimensional, for some \( \varepsilon \in \{0, 1\} \). Here there is no \textit{a priori} restriction on the eigenvalue \( \lambda \).

2. Another is that there is an integer \( k \) with

\[
V = V[k] \oplus V[k+2] \oplus V[k+4] \oplus \ldots,
\]

(all spaces nonzero), such that \( LV[k] = 0 \), in which case \( \lambda = \frac{k}{2}(1 - \frac{k}{2}) \).

3. Similarly there could be an integer \( k \) with

\[
V = V[k] \oplus V[k-2] \oplus V[k-4] \oplus \ldots,
\]

such that \( RV[k] = 0 \), in which case \( \lambda = -\frac{k}{2}(1 + \frac{k}{2}) \).

4. Finally, we could have

\[
V = V[2-k] \oplus V[4-k] \oplus \cdots \oplus V[k-4] \oplus V[k-2],
\]

in which case \( \lambda = \frac{k}{2}(1 - \frac{k}{2}) \) once again.

Now suppose \( G = \text{GL}_2(\mathbb{R}) \); the maximal compact subgroup of \( G \) is \( \text{O}(2) \). \( \text{O}(2) \) is a semidirect product of \( \mathfrak{so}(2) \) by an element of order 2, namely \( w = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \). If \( V \) is an irreducible admissible \((\mathfrak{g}, \text{O}(2))\)-module, then there are two possibilities: either the restriction of \( V \) to \((\mathfrak{g}, \text{SO}(2))\) is still irreducible, or else the restriction of \( V \) to \((\mathfrak{g}, \text{SO}(2))\) is the sum of two irreducibles which are swapped by \( w \). Note that \( w \) swaps \( V[k] \) with \( V[-k] \). There are the following possibilities:
1. \( V = \bigoplus_{k \equiv \varepsilon \pmod{2}} V[k] \), with each \( V[k] \not= 0 \) one-dimensional, for some \( \varepsilon \in \{0, 1\} \).

2. There exists an integer \( k \geq 1 \) with

\[
V = V[\pm k] \oplus V[\pm (k + 2)] \oplus \ldots ,
\]

and \( \lambda = \frac{k}{2}(1 - \frac{k}{2}) \).

3. There exists \( k \geq 2 \) with

\[
V = V[2 - k] \oplus V[4 - k] \oplus \cdots \oplus V[k - 4] \oplus V[k - 2].
\]

In fact all three possibilities occur. In the first case, \( V \) can be modeled on the space of \( K \)-finite vectors in an induced representation \( \pi(\chi_1, \chi_2) \). (It is possible to read off the eigenvalues of \( \Delta \) and \( z \) on \( V \) from the \( \chi_i \), but not necessary for us now.) This is the principal series.

In the second case, \( V \) can be modeled (up to twisting by a 1-dimensional character) on the space of \( K \)-finite vectors in a certain representation \( D_k \) of \( \text{GL}_2^+(\mathbb{R}) \) defined as follows: \( D_k \) is the space of holomorphic functions \( f \) on the upper half plane which satisfy

\[
\int_{\mathcal{H}} |f(z)|^2 y^k \frac{dx \, dy}{2} < \infty,
\]

with the action of \( g \in \text{GL}_2^+(\mathbb{R}) \) defined by \( f \mapsto f|_{g^{-1}, k} \). (If \( l \geq k \) has the same parity as \( k \), one can write down a nonzero vector \( f \in D_k[l] \) in terms of the coordinate \( w \) on the open unit disk, by \( f(w) = w^{(l-k)/2} \).) This is the discrete series (if \( k = 1 \) it is the limit of discrete series).

In the third case, \( V \) can be modeled (up to twisting by a 1-dimensional character) on the representation of \( G \) on the space of homogeneous polynomials of degree \( k - 2 \) in 2 variables.