MA 843 Assignment 2, due Sept. 24

September 13, 2013

1. Let $f: X \to Y$ be a finite morphism of degree $d$ between smooth projective curves over a field $K$.

   (a) Show that if $D$ is a divisor on $Y$ of degree $d$, then $\deg f^*(D) = d \deg D$.

   (b) If $D$ is a principal divisor on $Y$, show that $f^*(D)$ is also principal.

   (c) Show that if $D$ is a divisor on $X$ of degree $d$, then $\deg f_*(D) = \deg D$. (Very easy.)

   (d) If $D$ is a principal divisor on $X$, show that $f_*(D)$ is also principal.

   (The morphism $f$ induces a map of function fields $K(Y) \to K(X)$ which presents $K(X)$ as a finite extension of $K(Y)$. Thus there is a norm map $K(X)^* \to K(Y)^*$, which is key to this part.)

From this we can conclude that a correspondence from $X$ to $Y$ induces a well-defined map $\text{Pic}^0(X) \to \text{Pic}^0(Y)$.

2. Let $A$ be an abelian variety over $\mathbb{F}_q$. Let $P(T) \in \mathbb{Z}[T]$ be the characteristic polynomial of the $q$th power Frobenius map acting on $\ell$-adic Tate module of $A \,(\ell \nmid q)$. Let $\alpha_1, \ldots, \alpha_{2g}$ be its complex roots. Show that for all $n \geq 1$, $\#A(\mathbb{F}_{q^n}) = \prod_{i=1}^{2g} (1 - \alpha_i^n)$.

3. For abelian varieties $A$, $A'$, show that the dimension of the $\mathbb{Q}$-vector space $\text{Hom}(A, A') \otimes \mathbb{Q}$ is at most $4 \dim A \dim A'$. (If you like, assume that the base field is $\mathbb{C}$, so that $A$ and $A'$ are complex tori. In general, one uses Tate modules.)

4. Let $A$ be an abelian variety of dimension $g$ over a field $F$. $A$ is said to be of $\text{GL}_2$-type if the $\mathbb{Q}$-algebra $\text{End} \, A \otimes \mathbb{Q}$ contains a subfield $K$ of degree
$g$ over $\mathbb{Q}$. In this case, for every prime $\ell$ unequal to the characteristic of the base field of $A$, we get an action of $K_\ell := K \otimes_\mathbb{Q} \mathbb{Q}_\ell$ on the rational Tate module $V_\ell A = T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Show that $V_\ell A$ is a free $K_\ell$-module of rank 2. If $\lambda$ is a prime of $K$ above $\ell$, we get a surjective map $K_\ell \to K_\lambda$. Show that $V_\lambda = V_\ell A \otimes_{K_\ell} K_\lambda$ is a free $K_\lambda$-module of rank 2. We therefore get a Galois representation $\text{Gal}(\overline{F}/F) \to \text{Aut}_{K_\lambda} V_\lambda \approx \text{GL}_2 K_\lambda$.

5. Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a cuspidal eigenform of weight 2 for $\Gamma_0(N)$. Its coefficients lie in some totally real number field $K$. Let $S$ be the set of real embeddings of $K$, so that $\{ f^{\sigma} \}_{\sigma \in S}$ is the Galois orbit of $f$. Let $A$ be the modular abelian variety associated to $\{ f^{\sigma} \}_{\sigma \in S}$. Look up what the $L$-function of an abelian variety over $\mathbb{Q}$ is, and show that

$$L(A, s) = \prod_{\sigma \in S} L(f, s),$$

at least up to finitely many bad places. (It turns out that the $L$-functions are equal on the nose, but this is harder.)