1 The Deligne torus, and Hodge structures

Let $S$ be the real algebraic group $\text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$. Thus $S(\mathbb{R}) = \mathbb{C}^\times$.

If $V$ is a finite-dimensional real vector space, the data of a Hodge structure on $V$ is equivalent to the data of a morphism $h: S \to \text{GL}(V)$ of real algebraic groups. Let’s recall how this works: if such an $h$ is given, then for all pairs $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ we can define a subspace $V^{pq} \subset V\mathbb{C}$ by

$$V^{pq} = \left\{ v \in V\mathbb{C} \mid h(z) = z^{-p}z^{-q}v \quad z \in \mathbb{C}^\times \right\}.$$ 

Then the $V^{pq}$ are obviously disjoint, and $\overline{V^{pq}} = V^{qp}$. Further, the complex characters of $S$ are exactly the $z \mapsto z^{-p}z^{-q}v$, and any representation of $S$ on a complex vector space has to break up into such characters, so $V$ has to be the direct sum of the $V^{pq}$.

Conversely, if the $V^{pq}$ are given, then it is easy to define $h: S \to \text{GL}(V)$ in terms of the above formula.

We remark that $V$ is homogeneous of weight $k$ if and only if, for all $t \in \mathbb{R}^\times$, we have $h(t) = t^k I \in \text{GL}(V)$.

**Lemma 1.0.1.** Two Hodge structures on $V$ have the same combinatorial data (i.e. the dimensions of the $V^{pq}$) if and only if the corresponding morphisms $h: S \to \text{GL}(V)$ are conjugate.

**Proof.** If the two morphisms are conjugate by some $g \in \text{GL}(V)$, then $g$ induces an automorphism of $V\mathbb{C}$ carries the $V^{pq}$ for one Hodge decomposition onto the $V^{pq}$ for the other, so that these have the same dimension. Conversely, if $\{V_1^{pq}\}$ and $\{V_2^{pq}\}$ are two Hodge structures on $V$ with the same
combinatorial data, we can find \( g: V_\mathbb{C} \to V_\mathbb{C} \) which carries \( V_1^{pq} \) onto \( V_2^{pq} \) and which satisfies \( g(v) = \overline{g(v)} \). Since \( g \) commutes with complex conjugation, it descends to an automorphism of the real vector space \( V \); this automorphism carries one \( h \) onto the other.

In the situation of a family of Hodge structures which comes from family of smooth Kähler manifolds \( X \to B \), the combinatorial data of those Hodge structures (i.e., the dimensions of the \( V^{pq} \)) is constant. Therefore, the period map out of \( B \) can be said to have values in a single conjugacy class of morphisms \( h: S \to \text{GL}(V) \).

Deligne’s formalism of period domains generalizes this picture by replacing \( \text{GL}(V) \) by a general real algebraic group \( G \). (In the following, I am going to confuse the algebraic group \( G \) with the Lie group \( G(\mathbb{R}) \) a little bit.) Let \( X \) be a conjugacy class of maps \( h: S \to G \). We impose the following assumption:

\[ h(\mathbb{R}^\times) \text{ lies in the center of } G, \text{ for all (equivalently, one) } h \in X. \]

If \( h \in X \), then \( X = G/K \), where \( K \subset G \) is the stabilizer of \( h \), a Lie subgroup of \( G \). Thus \( X \) gets the structure of a smooth manifold. Our first order of business is to give \( X \) the structure of a complex manifold. To do this, it is necessary to give \( T_hX \) the structure of a complex vector space. We have \( T_hX = \text{Lie } G / \text{Lie } K \).

## 2 The complex structure on \( X = G/K \)

The real vector space \( L = \text{Lie } G \) is a \( G \)-module through the adjoint action \( \text{ad}: G \to \text{GL}(L) \)–this is the derivative of the action of \( G \to \text{Aut } G \) on itself through conjugation. Composing \( h: S \to G \) with \( \text{ad} \), we get a Hodge structure on \( L \). Note that since \( h(\mathbb{R}^\times) \) lies in the center of \( G \), \( \text{ad } h(\mathbb{R}^\times) \) is the identity on \( L \), which is to say that \( L \) is homogeneous of weight 0. Let \( L^{00} \subset L \) be the \((0,0)\)-part of this Hodge structure; that is, \( L^{00} \) is the space of vectors fixed by every \( h(z) \), \( z \in S \).

**Lemma 2.0.2.** \( L^{00} = \text{Lie } K \).

**Proof.** Since \( K \) is by definition the stabilizer of \( h \) in \( G \), we have \( (\text{ad } h(z))(k) = k \) for all \( k \in K \). Differentiating, we find that \( (\text{ad } h(z))(v) = v \) for all \( v \in \text{Lie } K \), so that \( \text{Lie } K \subset L^{00} \). Conversely, if \( v \in L^{00} \), then \( (\text{ad } h(z))(v) = v \);
exponentiating gives \((\text{ad } h(z))(\exp v) = \exp v\), so that \(\exp v \in K\), which means that \(v \in \text{Lie } K\).

**Lemma 2.0.3.** The natural inclusion \(L \rightarrow L_{\mathbb{C}}\) induces an isomorphism \(L/L^{00} \rightarrow L_{\mathbb{C}}/F^0L_{\mathbb{C}}\).

**Proof.** The image of \(L^{00}\) lands in the \((0,0)\)-part of \(L_{\mathbb{C}}\), which is certainly contained in \(F^0L_{\mathbb{C}}\). Thus \(L/L^{00} \rightarrow L_{\mathbb{C}}/F^0L_{\mathbb{C}}\) is well-defined.

If \(v \in L\) lies in \(F^0L_{\mathbb{C}} = \bigoplus_{p \geq 0} L^{pq}_{\mathbb{C}}\), then \(\bar{v} = v\) lies in \(\overline{F^0L_{\mathbb{C}}} = \bigoplus_{p \leq 0} L^{pq}_{\mathbb{C}}\). Thus \(v\) has to lie in the intersection of these two spaces, namely in the part where \(p = 0\). But since \(L\) has weight 0, the index \(q\) has to be 0 as well. Thus \(v \in L^{00}_{\mathbb{C}} \cap L = L^{00}\).

For surjectivity, let \(v \in L^{pq}_{\mathbb{C}}\). At least one of \(v\) and \(\bar{v}\) has to lie in \(F^0L_{\mathbb{C}}\), so that in the quotient \(L_{\mathbb{C}}/F^0L_{\mathbb{C}}\), \(v\) has the same class as \(v + \bar{v}\), a real vector. □

The previous two lemmas show that \(T_hX = L/L^{00} \cong L_{\mathbb{C}}/F^0L_{\mathbb{C}}\) has a natural complex structure. This gives \(X\) the structure of an almost complex manifold. To show that \(X\) really is a complex manifold, we are going to embed \(X\) in a flag variety \(\mathcal{F}\), which we already know is a complex manifold, and show that this embedding respects the almost complex structures on either side.

We choose a faithful representation \(G \rightarrow \text{GL}(V)\). Then for each \(h \in X\) we get a Hodge structure on \(V\). The combinatorial data of those Hodge structures doesn’t depend on \(h \in X\), though, since these are all conjugate. Let \(\mathcal{F}\) be the variety of filtrations of \(V_{\mathbb{C}}\) by subspaces of the appropriate dimension. Thus we get a map \(\phi: X \rightarrow \mathcal{F}\). Since the Hodge filtration determines the Hodge structure, this map is injective. Now let’s check it preserves complex structures at every \(h \in X\). First, we’ll give a convenient description of the tangent space to \(\mathcal{F}\) at \(\phi(h)\). The Hodge structure on \(V\) induces a Hodge structure on \(\text{End } V_{\mathbb{C}}\), with respect to which

\[
F^p \text{End } V_{\mathbb{C}} = \left\{ f \in \text{End } V_{\mathbb{C}} \left| f(F^qV_{\mathbb{C}}) \subset F^{p+q}V_{\mathbb{C}}, q \in \mathbb{Z}\right. \right\}.
\]

**Lemma 2.0.4.** \(T_\phi(h)\mathcal{F} = \text{End}(V_{\mathbb{C}})/F^0\text{End}(V_{\mathbb{C}})\).

**Proof.** The flag variety \(\mathcal{F}\) is the closed subspace of the product \(\prod_p \text{Grass}(b^p, V_{\mathbb{C}})\) consisting of those sequences \((F^pV_{\mathbb{C}})_p\) which are decreasing: \(F^pV_{\mathbb{C}} \supset F^{p+1}V_{\mathbb{C}}\).

We have identified the tangent space to \(\text{Grass}(b^p, V_{\mathbb{C}})\) at \(F^pV_{\mathbb{C}}\) with \(\text{Hom}(F^pV_{\mathbb{C}}, V_{\mathbb{C}}/F^pV_{\mathbb{C}})\).

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It is not hard to show that the tangent space to $\mathcal{F}$ is the subset of the direct sum $\bigoplus_p \text{Hom}(F^pV_C, V_C/F^pV_C)$ consisting of sequences $(f_p)$ satisfying the compatibility condition

$$f|_{F^{p+1}V_C} = f_{p+1} \pmod{F^pV_C}.$$ 

Now if $f \in \text{End}V_C$ is arbitrary, it gives rise to a sequence $(f_p)$ by $f_p = f|_{F^pV_C} \pmod{F^pV_C}$; this is easily seen to satisfy the compatibility condition. The condition that $f_p = 0$ for all $p$ is equivalent to the condition that $f(F^pV_C) \subset F^pV_C$ for all $p$, which is the same as the condition that $f \in F^0\text{End}V_C$. Thus $f \mapsto (f_p)$ is injective. A dimension count shows it is surjective (exercise).

The derivative of the action $G \to \text{GL}(V)$ is a map of Lie algebras $\alpha: L \to \text{End}V$. We have an adjoint action of $G$ on $\text{End}V$, and then $h: S \to G$ induces a Hodge structure on $\text{End}V$. Then the map $\alpha$ preserves Hodge structures, and in particular it takes $F^0$ to $F^0$. Thus $\alpha$ induces a map

$$L_C/F^0L_C \to \text{End}(V_C)/F^0\text{End}(V_C).$$

This is actually the derivative of $\phi: X \to \mathcal{F}$ at $h$. It is now manifest that $\phi$ preserves almost complex structure. This shows that $X$ really is a complex manifold. Note that the complex structure on $X$ doesn’t depend on the choice of $V$.

A nice consequence of this calculation is that whenever $V$ is a representation of $G$, we get a family of Hodge structures on $V$ parametrized by $X$ which varies holomorphically.

### 3 Griffiths transversality

We can now state what is probably the most important theorem of this lecture:

**Theorem 3.0.5.** The variation of Hodge structures on $V$ satisfies Griffiths transversality if and only if the Hodge structure on $L$ is of type $\{(−1,1), (0,0), (1,−1)\}$.

The condition on $L$ doesn’t depend on $V$. Thus if one $V$ satisfies Griffiths transversality, then they all must.
Proof. Recall the map $\alpha: L \to \text{End} V$ (the derivative of $G \to \text{GL}(V)$). We have seen that the derivative of $\phi: X \to \mathcal{F}$ at $h$ is the map

$$\alpha: L_C/F^0L_C \to \text{End}(V_C)/F^0 \text{End}(V_C)$$

induced from $\alpha$. Griffiths transversality is the condition that, for indices $p$, a vector $f$ in the image of this map should send $F^pV_C$ into $F^{p-1}V_C/F^pV_C \subset V_C/F^pV_C$. But this is precisely the condition that $f \in F^{-1} \text{End} V_C$. Since $\alpha$ is injective, this means that $L_C = F^{-1}L_C$. Since $L$ has weight 0, the only possible Hodge summands appearing in $L_C$ are $(1,-1), (0,0)$ and $(-1,1)$. \qed

4 Cartan involutions

The following is a digression on Cartan involutions which we’ll need when we talk about polarizable variations of Hodge structure.

Let $G$ be a real algebraic group, and let $\sigma$ be an involution of $G$; that is, an automorphism of order 2. The real form $G^\sigma$ is a real algebraic group whose $\mathbb{R}$-points are

$$G^\sigma(\mathbb{R}) = \left\{ g \in G(\mathbb{C}) \mid \sigma(g) = \overline{g} \right\}.$$ 

Generally, if $A$ is an $\mathbb{R}$-algebra, the $A$-points of $G^\sigma$ are

$$G^\sigma(A) = \left\{ g \in G(A \otimes \mathbb{C}) \mid \sigma(g) = \overline{g} \right\},$$

where $g \mapsto \overline{g}$ is interpreted as coming from the automorphism $1 \otimes z \mapsto 1 \otimes \overline{z}$ of $A \otimes \mathbb{C}$. Note that $G$ and $G^\sigma$ have the same base change to $\mathbb{C}$.

Example 4.0.6. For instance, let $\sigma$ be the automorphism of $G = GL_n$ which sends $g$ to $(g^t)^{-1}$. Then $G^\sigma = U(n)$, a compact group.

Example 4.0.7. For another example, let $G = O(n)$ and let $\sigma(g) = JgJ^{-1}$, where

$$J = \text{diag}(1, \ldots, 1, -1, \ldots, -1)$$

has signature $(r, s)$ ($r + s = n$). I claim that $G^\sigma = O(r, s)$. Indeed, let

$$\Psi = \text{diag}(1, \ldots, 1, i, \ldots, i),$$
so that $\Psi^2 = J$. We have

$$G^\sigma(\mathbb{R}) = \left\{ g \in G(\mathbb{C}) \mid gg^t = I, \, \bar{g} = JgJ^{-1} \right\}.$$ 

The map $g \mapsto \Psi g \Phi^{-1}$ maps $G^\sigma(\mathbb{R})$ isomorphically onto the set of $h$ which satisfy the conditions $hJh^t = J$ and $\bar{h} = h$, which is exactly $O(r, s)$. Note that $O(r, s)$ is only compact if $rs = 0$.

Note that a non-compact real group $G$ can have a compact form $G^\sigma$. (Pedantic note: compact means that $G^\sigma(\mathbb{R})$ is compact, and that $G^\sigma(\mathbb{R})$ meets every connected component of $G^\sigma(\mathbb{C}) = G^\sigma(\mathbb{C})$.) We say that $\sigma$ is a Cartan involution if $G^\sigma$ is compact.

**Theorem 4.0.8.** A connected group $G$ is reductive if and only if it admits a Cartan involution. In that case, any two Cartan involutions are conjugate.

(A reductive algebraic group is one whose unipotent radical is trivial.)

An obvious source of involutions on $G$ arises when you consider elements $C \in G(\mathbb{R})$ for which $C^2$ lies in the center of $G$. Then $\sigma(g) := CgC^{-1}$ is an involution of $G$. Call a representation $V$ of $G$ $C$-polarizable if there exists a $G$-invariant bilinear form $V \times V \to \mathbb{R}$ such that $\Psi(x, Cy)$ is symmetric and positive definite. The following theorem gives a condition for $\sigma$ to be a Cartan involution.

**Theorem 4.0.9.** Let $G$ be a real algebraic group, let $C \in G(\mathbb{R})$ be such that $C^2$ is central, and let $\sigma \in \text{Aut } G$ be conjugation by $C$. The following are equivalent.

1. $\sigma$ is a Cartan involution.

2. Every representation of $G$ is $C$-polarizable.

3. $G$ admits a faithful $C$-polarizable representation.

**Proof.** We’ll only sketch the proof that (1) and (3) are equivalent. First we observe that if $K$ is a compact Lie group, and $V$ is a complex representation of $K$, then $V$ admits a $K$-invariant positive definite hermitian form $H$. Indeed, let $H_0$ be any positive definite hermitian form on $V$, and let

$$H(v, w) = \int_{k \in K} H_0(kv, kw) dk,$$
where $dk$ is the Haar measure on $K$. Conversely, if $K$ is a Lie group, and $V$ is a faithful representation of $K$ admitting a $K$-invariant symmetric positive definite form $H$, then $K$ is compact, because it is a closed subset of $U(H)$, which is compact.

There is a version of this observation for real representations: if $K$ is a compact real Lie group, and $V$ is a real representation of $K$, then $V$ admits a $K$-invariant positive definite symmetric form $\psi$. Conversely, if $K$ is a Lie group, and $V$ is a faithful representation of $K$ admitting a $K$-invariant symmetric positive definite form, then $K$ is compact.

Assume (3). Let $V$ be a faithful representation of $G$ equipped with a $C$-polarization $\Psi$. Then $\Psi_C : V_C \times V_C \to C$ is a symmetric bilinear form which is $G(\mathbb{C})$-invariant. Let

$$H(u, v) = 2\Psi_C(u, \overline{v}),$$

so that $H$ is a positive definite hermitian form. Then the hermitian form $H$ is $G$-invariant in the sense that

$$H(gu, \overline{gv}) = H(u, v)$$

for $u, v \in V_C$. We claim that

$$H^\sigma(u, v) = H(u, Cv)$$

is a $G^\sigma(\mathbb{R})$-invariant positive definite hermitian form on $V_C$. First we check that $H^\sigma$ is hermitian:

$$H^\sigma(u, v) = \overline{H^\sigma(u, v)} = \overline{\Psi_C(\overline{u}, C\overline{v})} = \Psi_C(\overline{v}, C\overline{u}) = H(u, v).$$

Note that we used that $\Psi(u, Cv)$ was symmetric. Now we check $G^\sigma(\mathbb{R})$-invariance: for $g \in G^\sigma(\mathbb{R})$,

$$H^\sigma(gu, gv) = H(gu, Cgv) = H(gu, CgC^{-1}Cv) = H(gu, \overline{g}Cv) = H(u, Cv) = H^\sigma(u, v).$$

Finall, $H^\sigma$ is positive definite, since for $u \in V$ nonzero we have

$$H^\sigma(u, u) = H(u, Cv) = \Psi(u, Cu) > 0.$$

Thus $G^\sigma(\mathbb{R})$ admits a faithful representation admitting an invariant positive definite hermitian form, and so must be compact, so that we get (1).
Now assume (1), so that $G^\sigma(\mathbb{R})$ is compact. Let $V$ be any faithful (and real) representation of $G^\sigma(\mathbb{R})$; then there exists a $G^\sigma(\mathbb{R})$-invariant positive definite symmetric form $\psi$ on $V$. Let $\Psi: V \times V \to \mathbb{R}$ be

$$
\Psi(u, v) = \psi(u, C^{-1}v).
$$

Then $\Psi(u, Cv) = \psi(u, v)$ is positive definite and $G(\mathbb{R})$-invariant (since it is real-valued and invariant by $G(\mathbb{C})$).

\[\square\]

5 Variations of polarizable Hodge structures

Let $h: S \to \text{GL}(V)$ be a Hodge structure of weight $k$ on a real vector space $V$, so that we get a decomposition $V_\mathbb{C} = \bigoplus V^{pq}$. A polarization on $V$ is a bilinear form $\Psi: V \times V \to \mathbb{R}$ which turns $V$ into a polarized Hodge structure. Recall that this means that $\Psi$ is alternating or symmetric as $k$ is odd or even, and if we let $H: V_\mathbb{C} \times V_\mathbb{C} \to \mathbb{R}$ be the associated Hermitian form

$$
H(\alpha, \beta) = i^k \Psi(\alpha, \overline{\beta}),
$$

then the $V^{pq}$ are orthogonal with respect to $H$, and $H$ has sign $i^{p-q-k}$ when restricted to $V^{pq}$. (In earlier lectures there was an additional factor of $(-1)^{k(k-1)/2}$, in accordance with Voisin’s book. Since $k$ is fixed throughout the discussion, we can ignore this difference in sign.)

In terms of $h$, this means the following.

**Lemma 5.0.10.** A bilinear form $\Psi: V \times V \to \mathbb{R}$ is a polarization if and only if it satisfies the following two conditions:

$$
\Psi(h(z)v, h(z)w) = |z|^{-k} \Psi(v, w),
$$

for all $v, w \in V, z \in S(\mathbb{R}) = \mathbb{C}^\times$, and

$$
\Psi(v, h(i)v) \text{ is symmetric and positive definite}
$$

This elegant characterization is a small advertisement for the “$h$” point of view. If $\mathbb{R}(n)$ denotes the vector space $\mathbb{R}$ together with the Hodge structure $z \mapsto |z|^n$, then in the context of the lemma we have a morphism of Hodge structures $\Psi: V \otimes V \to \mathbb{R}(-k)$ (it commutes with the $h$s).
Proof. We’ll do the “if” direction and leave the converse as an exercise. Since $V$ has weight $k$, $h(-1) = (-1)^k I_V$. The symmetry of $\Psi(v, h(i)w)$ means that

$$
\Psi(w, v) = (-1)^k \Psi(w, h(i)^2 v) = (-1)^k \Psi(h(i)v, h(i)w) = (-1)^k \Psi(v, w),
$$

so that $\Psi$ is alternating or symmetric as $k$ is odd or even. Now let $H(v, w) = i^k \Psi(v, \overline{w})$ as usual. We claim that the $V^{pq}$ are orthogonal with respect to $H$. Let $v \in V^{pq}$ and $w \in V^{rs}$. We have

$$
|z|^{-k} H(v, w) = H(h(z)v, h(z)w) = H(z^{-p} z^{-q}, z^{-r} \overline{z}^{-s} w) = \Psi(z^{-p} z^{-q}, z^{-s} \overline{z}^{-r} \overline{w}) = z^{-p-s} \overline{z}^{-q-r} H(v, w)
$$

This means that $H(v, w) = 0$ unless $r = p$ and $s = q$. Thus the $V^{pq}$ are orthogonal with respect to $H$.

The condition that $\Psi(v, h(i)w)$ be positive definite means that $\Psi(v, h(i)\overline{w}) = i^{-k} H(v, h(i)w)$ is a positive definite hermitian form. For $v \in V^{pq}$ nonzero we have

$$
0 < i^{-k} H(v, h(i)v) = i^{p-q-k} H(v, v),
$$

and therefore $H(v, v)$ has the correct sign.

We return to Deligne’s setting. Let $G$ be a real algebraic group, and let $X$ be the conjugacy class of maps $S \to G$. Keep the assumption that each $h \in X$ is central. Let $V$ be a faithful representation of $G$, so that each point of $X$ gives a Hodge structure on $V$. Call $V$ polarizable if there exists a bilinear form $\Psi : V \times V \to \mathbb{R}$, such that each $h \in X$ determines a Hodge structure on $V$ which is polarized with respect to $\Psi$.

A little explanation is in order. $V$ might not be homogeneous of a particular weight, but (because of our assumption on $h(R^\times)$) it does break up into $G$-invariant summands $V_k$, which are homogenous of weight $k$. Polarizability now means that for each $k$, $V_k$ admits a bilinear form (symmetric or alternating, as $k$ is even or odd), such that each $h \in X$ determines a polarized Hodge structure on $V_k$.

Remarkably, the condition that $V$ be polarizable is intrinsic to $X$. That is, if one $V$ is polarizable, then they all are. This follows from the following theorem.
Theorem 5.0.11. Let $G_1$ be the smallest subgroup of $G$ through which all the $h \in X$ factor. $V$ is polarizable if and only if the following conditions hold: $G_1$ is reductive, and for some $h \in X$ (equivalently, all of them), conjugation by $h(i)$ is a Cartan involution on the adjoint group $G_1^{\text{ad}}$.

We remark that if $G$ is connected then its adjoint group is simply what you get when you mod out by the center.

Let's examine the criterion that $V$ be polarizable. In what follows, we assume that $V$ is homogeneous of weight $k$. Let $\Psi : V \times V \to \mathbb{R}$ be a polarization. It will be useful to introduce the subgroup $G_2 \subset G_1$, defined as the smallest subgroup containing $h(U^1)$ for each $h \in X$ (where $U^1 \subset \mathbb{C}^\times$ is the subgroup of norm 1 elements). For all $z \in U^1$, and all $v \in V^{pq}$, we have (by the last lemma) $\Psi(h(z)v, h(z)w) = \Psi(v, w)$, so that $\Psi$ is $G_2$-invariant. We also have that $\Psi(v, h(i)w)$ is symmetric and positive definite for all $h \in X$. This means that $V$ is a representation of $G_2$ admitting an $h(i)$-polarization. From this we get that $h(i)$ is a Cartan involution on $G_2$.

For its part, the group $G_1$ is generated by $G_2$ together with the elements $h(t)$, where $h \in X$ and $t \in \mathbb{R}^\times$. The latter elements are central in $G$. From here one can show that $G_1$ and $G_2$ have the same adjoint group. This gives one direction of the theorem. For the other, suppose that conjugation by $h(i)$ is a Cartan involution on $G_1^{\text{ad}} = G_2^{\text{ad}}$. Since $G_2$ is generated by compact subgroups, its center is compact (?), so that an involution on $G_2$ is Cartan if and only if it is Cartan on $G_2^{\text{ad}}$. Thus conjugation by $h(i)$ is Cartan on $G_2$. Let $V$ be a faithful representation of $G$, and let $\Psi$ be an $h(i)$-polarization on $V$ with respect to $G_2$. By definition, $\Psi$ is $G_2$-invariant. This means that $V$ is a polarizable.

6 Hermitian symmetric domains

We are thus led to study pairs $(G, X)$, where $G$ is a real algebraic group and $X$ is a conjugacy class of nontrivial morphisms $h : S \to G$, which satisfy the following conditions. (If an $h \in X$ satisfies one of these conditions, then they all do.)

1. $h(\mathbb{R}^\times)$ is central.

2. The Hodge structure on Lie $G$ induced by $\text{ad} \circ h$ is of type $\{(1, -1), (0, 0), (-1, 1)\}$.

3. Conjugation by $h(i)$ is a Cartan involution on the adjoint group of $G_1$. 
(Actually, Deligne wants to consider only connected $X$, in which case $X$ is a conjugacy class under the neutral component of $G(\mathbb{R})$.)

Let’s first examine the case that $G = G_1$ is a simple adjoint group. Let $h \in X$. Let $K \subset G(\mathbb{R})$ be the centralizer of $X$, so that $X = G(\mathbb{R})/K$ (or a connected component thereof). Let $\sigma = \text{ad}(i)$, so that $G^\sigma$ is compact. We have $K \subset G^\sigma$ is a closed subgroup, so that $K$ is compact as well. We can give $X$ a $G(\mathbb{R})$-invariant Riemannian metric, by putting a $K$-invariant symmetric positive definite form on $T_h X$, and using $G$ to translate this to all of $X$.

The tangent space to $X$ at $h$ is $\text{Lie}(G)/\text{Lie}(G)^0$, and $\text{ad}(i)$ acts on it by $-1$. This means that for all $h \in X$ there exists an involution $\sigma_h$ of $X$ (namely, conjugation by $h(i)$) which has $p$ as an isolated fixed point, and whose derivative at $p$ is the scalar $-1$. It can be shown that $\sigma_h$ is an isometry of $X$.

On the other hand, $X$ is a complex manifold. The complex and Riemannian structures combine to give $X$ the structure of a hermitian manifold, which happens to have an involutive symmetry at each point.

**Definition 6.0.12.** A hermitian symmetric space is a hermitian manifold $X$ with these properties:

1. The automorphism group of $X$ acts transitively.
2. At each $p \in X$, there exists an involution $s_p$ of $X$ which has $p$ as an isolated fixed point.

$X$ is a hermitian symmetric domain if in addition it is noncompact.

**Theorem 6.0.13.** The manifolds $X$ constructed from pairs $(G, X)$ as above are exactly the hermitian symmetric domains.

### 7 Classification of hermitian symmetric domains in terms of simple complex algebraic groups

We would like to classify the pairs $(G, X)$, where $G$ is a real algebraic group and $X$ is a conjugacy class of maps $h: S \to G$ satisfying the conditions (1)–(3) above. In our classification, we’re going to make two simplifying assumptions on $G$:
• $G$ is simple (no connected normal subgroups other than 1 and $G$). Eg, SL($n$).

• $G$ is adjoint (the adjoint map $\text{ad}: G \to \text{GL}(\text{Lie } G)$ is a faithful representation of $G$.

Examples of simple adjoint groups: $\text{PSL}(n) = \text{SL}(n)/\{\pm I\}$, $\text{PSp}(n) = \text{Sp}(n)/\{\pm I\}$, $\text{PSU}(p,q)$ (the special unitary group $\text{SU}(p,q)$ modulo its center), $\text{SO}(p,q)/\{pmI\}$.

Now the conditions on $h: S \to X$ can be restated this way:

1. With respect to $h$, the adjoint representation $\text{Lie } G$ is of type $\{(1,-1),(0,0),(-1,1)\}$.

2. Conjugation by $h(i)$ is a Cartan involution of $G$.

3. $h$ is nontrivial.

Condition (1) above means that $\text{ad} \circ h(\mathbb{R}^\times)$ acts trivially on $\text{Lie } G$. Since $\text{ad}$ is a faithful representation, this means that $h(\mathbb{R}^\times) = 1$. Also, given (1) and (2), the condition (3) is equivalent to saying that $G$ is noncompact. Indeed, if $h$ is trivial, then by (2) we have that the identity is a Cartan involution, which means that $G$ is compact. Conversely, if $G$ is compact, then the uniqueness of the Cartan involution shows that $\text{ad} \circ h(i)$ is the identity. But by (1), $\text{ad} \circ h(i)$ acts on $\text{Lie } G/(\text{Lie } G)_{00}$ by $-1$. This shows that $\text{Lie } G = (\text{Lie } G)_{00}$, and therefore that (for all $z$) $\text{ad } h(z)$ is the identity. Since $\text{ad}$ is faithful, $h$ is trivial.

We will now classify pairs $(G, X)$, where $G$ is a simple adjoint group over $\mathbb{R}$ and $X$ is a conjugacy class of $h$ satisfying (1)–(3) above. Of course, it’s easier to classify objects over $\mathbb{C}$ than over $\mathbb{R}$. So let’s examine what happens when we base change everything to $\mathbb{C}$.

Let $\mathbb{G}_m$ be the multiplicative group, considered as an algebraic group over $\mathbb{C}$. We have an isomorphism of complex algebraic groups $S_\mathbb{C} \to \mathbb{G}_m \times \mathbb{G}_m$ which sends $(a,b)$ to $(a + bi, a - bi)$. Let $\mathbb{G}_m \to S_\mathbb{C}$ be the inclusion onto the first coordinate. If $h: S \to G$ is a morphism, let $\mu_h: \mathbb{G}_m \to G_\mathbb{C}$ be the composite

$$\mu_h: \mathbb{G}_m \to S_\mathbb{C} \to^h G_\mathbb{C}.$$ 

Then the $G(\mathbb{C})$-conjugacy class of $\mu_h$ only depends on the $G(\mathbb{R})$-conjugacy class of $h$. If $V$ is a (real) representation of $G$, so that $h$ determines a Hodge structure on $V$, then $\mu_h$ is characterized by

$$\mu_h(z)v = z^pv, \quad z \in \mathbb{C}^\times, \quad v \in V^{pq} \subset V_\mathbb{C}$$

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The characters of the algebraic group $G_m$ are the power maps $z \mapsto z^n$, for $n \in \mathbb{Z}$. Any representation $G_m \to \text{GL}(V)$ ($V$ a finite-dimensional $\mathbb{C}$-vector space) breaks up to characters of this type.

**Lemma 7.0.14.** Assume that $h: G_m \to G$ satisfies property (1) (that $h(\mathbb{R})$ is central in $G$). The morphism $h$ satisfies property (2) above (that $\text{Lie } G$ is of type $\{(1, -1), (0, 0), (-1, 1)\}$ with respect to the Hodge structure induced by $\text{ad} \circ h$) if and only if representation $\text{ad} \circ \mu_h$ of $G_m$ on $\text{Lie } G^C$ contains only characters of type $z, 1, z^{-1}$.

**Proof.** $\text{Lie } G^C$ is a direct sum of spaces $(\text{Lie } G^C)^{pq}$, where $p + q = 0$ (condition (1) guarantees that the Hodge structure on $\text{Lie } G$ has weight $0$). On $(\text{Lie } G^C)^{pq}$, $\mu_h(z)$ acts by $z^p$. Thus $p$ takes only values $1, 0, -1$ if and only if $(p, q)$ takes the values $(1, -1), (0, 0), (-1, 1)$. $\square$

The following theorem tells how to recover the pair $(G, X)$ from its “complexification”.

**Theorem 7.0.15.** Let $H$ be a simple complex adjoint algebraic group. There is a bijection between the following two sets:

- Pairs $(G, X)$, where $G$ is a real form of $H$, and $X$ is a conjugacy class of maps $h: S \to G$ satisfying (1)–(3),

- $H(\mathbb{C})$-conjugacy classes of nontrivial maps $\mu: G_m \to H$ such that only $z, 1, z^{-1}$ appear in $\text{Lie } H$.

**Proof.** By the lemma, the complexification of a $(G, X)$ satisfying (1)–(3) gives such a $\mu$. Now suppose $\mu$ is given. Since $H$ is simple, it is reductive and therefore admits a compact real form $G^*$ of $H$. It is not hard to see that $h \mapsto \mu_h$ is a bijection between $G^*(\mathbb{R})$-conjugacy classes of $h: S \to G^*$ (which are trivial on $\mathbb{R}^*$) and $H(\mathbb{C})$-conjugacy classes of $\mu: G_m \to G^*_C = H$. Thus we have a map $h: S \to G^*$, defined over $\mathbb{R}$.

Let $G$ be the real form of $G^*$ corresponding to the involution $g \mapsto h(i)gh(i)^{-1}$. Then conjugation by $h(i)$ is a Cartan involution on $G$. Via the lemma, the condition on $\mu$ translates into the required condition on $h$. $\square$

If $G$ is a simple adjoint algebraic group over $\mathbb{C}$, what are the $\mu: G_m \to G$ which satisfy the condition in Thm. 7.0.15?

Let’s recall some basics from the theory of simple complex Lie groups. Let $\mathfrak{g} = \text{Lie } G$, a simple complex Lie algebra. There is a nondegenerate bilinear
form (the Killing form) $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$, defined by $(x, y) = \text{tr}((\text{ad } x)(\text{ad } y))$, where $\text{ad } x : \mathfrak{g} \rightarrow \mathfrak{g}$ is $y \mapsto [x, y]$. (In fact, $B$ being nondegenerate is equivalent to $\mathfrak{g}$ being semisimple—this is Cartan’s criterion.)

Let $T$ be a maximal torus of $G$. Let

$$X(T) = \text{Hom}(T, \mathbb{G}_m)$$
$$Y(T) = \text{Hom}(\mathbb{G}_m, T)$$

be the character and cocharacter groups of $T$. These are written additively. Then $X(T)$ and $Y(T)$ are dual under the pairing $X(T) \times Y(T) \rightarrow \text{Hom}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$. The Lie algebra $\mathfrak{g}$ admits an action of $T$ via $\text{ad}$, and must therefore break up into characters of $T$. The nontrivial characters $\alpha \in X(T)$ which appear in $\mathfrak{g}$ are the roots of $G$. Let $\Phi$ be the set of roots. We have the following decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ is the $\alpha$-eigenspace:

$$\mathfrak{g}_\alpha = \left\{ x \in \mathfrak{g} \mid (\text{ad } t)(x) = \alpha(t)x, \ t \in T \right\}.$$ 

The restriction of the Killing form to $\mathfrak{t}$ is nondegenerate, so that it supplies an isomorphism between $\mathfrak{t}$ and $\mathfrak{t}^\vee$, its $\mathbb{C}$-linear dual.

We have a map $X(T) = \text{Hom}(T, \mathbb{G}_m) \rightarrow \text{Hom}_\mathbb{C}(\mathfrak{t}, \text{Lie } \mathbb{G}_m) = \mathfrak{t}^\vee$, so we can think of $\Phi$ as a subset of $\mathfrak{t}^\vee$. From this point of view,

$$\mathfrak{g}_\alpha = \left\{ x \in \mathfrak{g} \mid [H, x] = \alpha(H)x, \ H \in T \right\}.$$ 

We claim that $\Phi$ spans $\mathfrak{t}^\vee$ as a complex vector space. Otherwise there would be a nonzero $H \in \mathfrak{t}$ for which $\alpha(H) = 0$ for all roots $\alpha$. But then $\text{ad } H$ kills $\mathfrak{g}_\alpha$ for all $\alpha$, so that $\text{ad } H$ kills all of $\mathfrak{g}$, and then we would have $H = 0$.

Let $V = \mathfrak{t}^\vee \otimes \mathbb{R}$. It so happens that the Killing form restricts to a positive definite form on $V$. The roots $\Phi$ span $V$, and have the following properties:

1. The only scalar multiples of $\alpha \in \Phi$ within $\Phi$ are $\pm \Phi$.
2. $\Phi$ is invariant under the reflections $s_\alpha$, defined by

$$s_\alpha(\beta) = \beta - 2\frac{(\alpha, \beta)}{(\alpha, \alpha)}\alpha.$$
3. If \( \alpha, \beta \in \Phi \), then \( 2(\alpha, \beta)/(\alpha, \alpha) \) is an integer. (The projection of \( \alpha \) onto the line through \( \beta \) is a half-integer multiple of \( \beta \).)

This means that \( \Phi \) is a root system.

It is possible to choose a set of positive roots \( \Phi^+ \subset \Phi \) which contains exactly one of \( \alpha, -\alpha \) for each \( \alpha \in \Phi \), and which has the property that for all \( \alpha, \beta \in \Phi^+ \) for which \( \alpha + \beta \in \Phi \), we have \( \alpha + \beta \in \Phi^+ \). (The choice of \( \Phi^+ \) is equivalent to the choice of Borel subgroup of \( G \) containing \( T \). The Weyl group of \( T \) acts transitively on the set of such Borels.)

Within \( \Phi^+ \) we have the set of simple roots \( B \): these are the roots which are not the sum of two other positive roots. Every positive root can be written uniquely as \( \sum_{\alpha \in B} m_\alpha \alpha \), where \( m_\alpha \geq 0 \). \( V \) is spanned by \( B \).

The Dynkin diagram of the root system has one vertex for each simple root \( \alpha \). Two roots are joined by \( \{0, 1, 2, 3\} \) edges, as the angle between \( \alpha \) and \( \beta \) is \( \{\pi/2, 2\pi/3, 3\pi/4, 5\pi/6\} \). In the latter two cases, an arrow is drawn which points to the shorter vector.

The Weyl chamber corresponding to \( B \) is
\[
\left\{ v \in V \left| (\alpha, v) \geq 0 \quad \alpha \in B \right. \right\};
\]
this is an intersection of half-planes in \( V \). The Weyl group permutes the Weyl chambers simply transitively.

The last fact we need concerns the highest root:

**Theorem 7.0.16.** There exists a unique root \( \alpha_+ = \sum_{\alpha \in B} n_\alpha \alpha \), such that if \( \sum_\alpha m_\alpha \alpha \) is any other root, then \( n_\alpha \geq m_\alpha \) for all \( \alpha \). (In particular \( n_\alpha \geq 1 \) for all \( \alpha \).

**Theorem 7.0.17.** Conjugacy classes of nontrivial minuscule \( \mu: \mathbb{G}_m \to G \) are in correspondence with simple roots \( \alpha \) with \( n_\alpha = 1 \).

If \( \mu: \mathbb{G}_m \to G \) is a cocharacter, then the image of \( \mu \) is contained in some maximal torus; after replacing \( \mu \) with a conjugate we assume that \( \mu: \mathbb{G}_m \to T \) is actually a cocharacter: \( \mu \in Y(T) \). Finally, we can translate \( \mu \) by an element of the Weyl group to assume that
\[
(\alpha, \mu) \geq 0, \quad \alpha \in B
\]
The condition that only the characters \( z, 1, z^{-1} \) appear in \( \text{Lie} \, G \) is equivalent to the condition that \( (\alpha, \mu) \in \{1, 0, -1\} \) for all roots \( \alpha \in \Phi \). Combining this
with the above, we find that \((\alpha, \mu) \in \{0, 1\}\) for all \(\alpha \in B\). Assuming that \(\mu\) is nontrivial, we have that \((\alpha_0, \mu) = 1\) for at least one \(\alpha_0 \in B\). We have 
\[
(\alpha_+, \mu) = \sum_\alpha n_\alpha (\alpha, \mu) = 1,
\]
so that \(n_{\alpha_0} = 1\). Note that this forces \((\alpha, \mu) = 0\) for all other simple roots \(\alpha \neq \alpha_0\).

Conversely, let \(\alpha_0 \in B\) be a simple root with \(n_{\alpha_0} = 1\). Let \(\mu \in Y(T)\) be defined by the conditions \((\alpha_0, \mu) = 1\) and \((\alpha, \mu) = 0\) for all other simple roots \(\alpha\). We claim that \((\beta, \mu) \in \{1, 0, -1\}\) for all roots \(\beta\). It is enough to assume that \(\beta \in \Phi^+\), and then \(\beta\) is a sum of simple roots: 
\[
\beta = \sum_\alpha m_\alpha \alpha,
\]
with \(m_\alpha \geq 0\). Then \((\beta, \mu) = m_\alpha \leq n_\alpha = 1\), so that \(m_\alpha \in \{0, 1\}\).

The nodes of the Dynkin diagram corresponding to simple roots \(\alpha\) with \(n_\alpha = 1\) are called special nodes.

8 Examples

8.1 The \(A_n\) root system

Let \(V \subset \mathbb{R}^{n+1}\) be the subspace of vectors whose coordinates sum to 0. The \(A_n\) root system is 
\[
\Phi = \left\{ v \in V \cap \mathbb{Z}^{n+1} \middle| \|v\| = \sqrt{2} \right\}.
\]
For the simple roots we can take \(\alpha_i = e_i - e_{i+1}, i = 1, \ldots, n\). The positive roots take the form 
\[
e_i - e_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1},
\]
for \(j > i\). Thus the highest root is 
\[
\alpha_+ = \alpha_1 + \cdots + \alpha_n = e_1 - e_{n+1}.
\]
We see that every node of \(A_n\) is special. Let \(1 \leq p \leq n\).

\(A_n\) is the root system corresponding to the simple group \(\text{PGL}_n\). If \(\mu\) is the cocharacter corresponding to the (special) node \(\alpha_p\), so that 
\[
\mu(z) = \text{diag}(1, 1, \ldots, 1, z, z, \ldots, z)\] with \(p\) 1s,
then the real form corresponding to \(\mu\) is \(G = \text{PU}(p, q)\), and the corresponding hermitian symmetric domain is \(U(p, q)/U(p) \times U(q)\).

8.2 The \(B_n\) root system

The \(B_n\) root system is 
\[
\Phi = \left\{ v \in \mathbb{Z}^n \middle| \|v\| = 1 \text{ or } \sqrt{2} \right\}.
\]
For the simple roots we can take \(\alpha_i = e_i - e_{i+1}\) for \(i = 1, \ldots, n-1\) and \(\alpha_n = e_n\). Then the highest
root is
\[ \alpha_+ = e_1 + e_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n. \]
Thus there is only one special node, namely the node corresponding to \( \alpha_1 \).

\( B_n \) is the root system corresponding to the simple group \( SO(2n+1) \). The cocharacter corresponding to \( \alpha_1 \) is
\[ \mu(z) = \left( \begin{array}{cc} (z + z^{-1})/2 & (z - z^{-1})/2i \\ -(z - z^{-1})/2i & (z + z^{-1})/2 \end{array} \right) \bigoplus I_{2n-1}. \]

The real form corresponding to \( \mu \) is \( G = SO(2, 2n-1) \), and the corresponding hermitian symmetric domain is \( O(2, 2n-1)/O(2) \times O(2n-1) \).

In fact, the only root systems whose Dynkin diagrams admit special nodes are those of type \( A_n, B_n, C_n, D_n, E_6 \) and \( E_7 \). The remaining root systems (those of type \( F_4, E_8 \) and \( G_2 \)) do not have associated hermitian symmetric domains (and therefore no Shimura varieties).