The Baily-Borel compactification

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Let $G$ be a simple algebraic group defined over $\mathbb{Q}$, and suppose that $X = G(\mathbb{R})/K$ is a hermitian symmetric domain. (Note that this already places restrictions on $G_\mathbb{R}$—the simple factors of its Lie algebra have to be of type $A_n$, $B_n$, $C_n$, $D_n$, $E_6$ or $E_7$.) For instance, $G$ could be $\text{SL}_2$ and $X$ could be $\mathcal{H}$, the upper half-plane. Or, $G$ could be $\text{Sp}_{2g}$ and $X$ could be $G(\mathbb{R})/S(O(2) \times O(n-2))$. In those cases, $G_\mathbb{R}$ was simple, but we do not want to exclude situations where $G_\mathbb{Q}$ is simple but $G_\mathbb{R}$ is not.

We can try to give meaning to the expression $G(\mathbb{Z})$. For instance, one can always embed $G$ into $GL_n$ (as algebraic groups over $\mathbb{Q}$), and then set $G(\mathbb{Z}) = GL_n(\mathbb{Z}) \cap G(\mathbb{Q})$. Call a subgroup $\Gamma \subset G(\mathbb{Q})$ arithmetic if the intersection $\Gamma \cap G(\mathbb{Z})$ has finite index in both $\Gamma$ and $G(\mathbb{Z})$. The theorem of Baily-Borel states that for an arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$, the quotient $Y = \Gamma \backslash X$ is a quasi-projective variety. This is proved by constructing a compactification $\overline{Y}$ (as a complex analytic space) and an isomorphism of $\overline{Y}$ with a closed subset of some projective space. By GAGA, $\overline{Y}$ is a projective variety, and therefore $Y$ is quasi-projective.

1 Basic examples: Modular curves and Hilbert modular surfaces

1.1 Modular curves

Suppose $G = \text{SL}_2/\mathbb{Q}$. Let $\Gamma \subset \text{SL}_2(\mathbb{Q})$ be an arithmetic subgroup. Then $Y = \Gamma \backslash \mathcal{H}$ is a can be compactified by adding finitely many cusps. Recall that a cusp for $\Gamma$ is a coset in $\Gamma \backslash \mathbb{P}^1$. Let $\overline{Y}$ be the topological space obtained
by adding the cusps of $\Gamma$ to $Y$. This is given the following structure of a complex manifold by declaring that a basis of neighborhoods of the cusp $\infty$ is \( \left\{ z \in \mathcal{H} \mid \text{Im} z > B \right\} \cup \{ \infty \} \) (for $B > 0$); for large enough $B$ this set is in bijection with an open subset of $\mathbb{C}$ via $z \mapsto e^{2\pi i/N}$, where $N$ is the index of $\Gamma \cap \left( \frac{1}{1} \mathbb{Z} \frac{1}{1} \right)$ in $\left( \frac{1}{1} \mathbb{Z} \frac{1}{1} \right)$. Then $Y$ is a compact Riemann surface, and therefore a projective curve.

Already we can see a connection between cusps and parabolic subgroups of $G$. A parabolic subgroup $P$ of an algebraic group $G$ is one for which $G/P$ is a projective variety; in the case of $G = \text{SL}_2 / \mathbb{Q}$, the parabolic subgroups are all conjugate to the subgroup $P$ of upper-triangular matrices, and $G(\mathbb{Q})/P(\mathbb{Q}) = \mathbb{P}^1(\mathbb{Q})$ is the set of all parabolic subgroups of $G(\mathbb{Q})$.

Consider the case of $\Gamma = \text{SL}_2(\mathbb{Z})$ (and brush aside issues arising from the elliptic fixed points). We know that $Y = \mathbb{P}^1$ via the $j$-invariant, but we can also see this using modular forms. The ring of modular forms of $\Gamma$ is a graded ring isomorphic to $\mathbb{C}[x, y]$ (where $x$ has weight 4 and $y$ has weight 6). Then we can form $\text{Proj} \mathbb{C}[x, y]$, which is $\mathbb{P}^1$. In other words: for each even $k$, we get a map $Y \to \mathbb{C}^N$ by $z \mapsto [f_0(z) : \cdots : f_N(z)]$, where $f_0, \ldots, f_N$ is a basis for the space of weight $k$ forms. For $k = 12$ this morphism separates points and tangent directions: $z \mapsto [E_4(z) : E_4(z)^3 - E_6(z)^2] = j(z)$ (modulo some constants). This is going to be the pattern in general: if we can find enough automorphic forms, we can find an embedding in projective space.

### 1.2 Hilbert modular surfaces

Now suppose $F = \mathbb{Q}(\sqrt{d})$ is a real quadratic field, with real embeddings $i_1$ and $i_2$. Let $G = \text{Res}_{F/\mathbb{Q}}(\text{SL}_2)$, so that $G(\mathbb{R}) = \text{SL}_2 \times \text{SL}_2$; the corresponding hermitian symmetric space is $X = \mathcal{H} \times \mathcal{H}$. This has an action of $G(\mathbb{Q}) = \text{SL}_2(\mathcal{O}_F)$, by $\gamma(z_1, z_2) = (i_1(\gamma)z, o_2(\gamma)z_2)$. For now let’s consider the case of the arithmetic subgroup $\Gamma = \text{SL}_2(\mathcal{O}_F)$. Let $Y = \Gamma \setminus X$. It turns out that $\Gamma \setminus Y$ can be given a moduli interpretation in terms of polarized abelian varieties endowed with an action of an order in $F$.

Indeed, for $z = (z_1, z_2) \in X$, put

\[
\Lambda_z = \left\{ i_1(\alpha)z_1 + i_1(\beta), i_2(\alpha)z_2 + i_2(\beta) \mid (\alpha, \beta) \in \mathcal{O}_F \times \mathcal{O}_F \right\},
\]

so that $\Lambda_z \approx \mathcal{O}_F \times \mathcal{O}_F$ is a subgroup of $\mathbb{C}^2$ of rank 4. In fact it is a lattice
in $\mathbb{C}^2$, because $\Lambda_z \otimes \mathbb{R} = (\mathbb{R}z_1 + \mathbb{R}) \times (\mathbb{R}z_2 + \mathbb{R}) = \mathbb{C}^2$ (since $z_1, z_2 \in \mathcal{H}$). Let $A_z = \mathbb{C}^2/\Lambda_z$. For $r \in F$, define

$$E_r : \mathcal{O}_F^2 \times \mathcal{O}_F^2 \rightarrow \mathbb{Q}$$

by

$$E_r((\alpha_1, \beta_1), (\alpha_2, \beta_2)) = \text{tr}_{F/\mathbb{Q}}(r(\alpha_1\beta_2 - \alpha_2\beta_1))$$

Then $E_r$ is $\mathbb{Q}$-bilinear alternating, and has integer values if $r$ belongs to $D_{F/\mathbb{Q}}^{-1}$ (the inverse to the different ideal). We have $E_r(\gamma \cdot \alpha, \beta) = E_r(\alpha, \gamma \cdot \beta)$ for all $\gamma \in \mathcal{O}_L$, where $\gamma \cdot (\alpha_1, \alpha_2) = (\gamma \alpha_1, \gamma \alpha_2)$. Consider $E_r$ as a form on $\Gamma_z$, and extend $\mathbb{R}$-linearly to get an alternating form $E_{r,z}$ on the real vector space $\mathbb{C}^2$.

If we let $H_{r,z}$ be the hermitian form on $\mathbb{C}^2$ defined by

$$H_{r,z}((x_1, x_2), (y_1, y_2)) = \frac{x_1 y_1 i_1(r)}{\text{im} z_1} + \frac{x_2 y_2 i_2(r)}{\text{im} z_2},$$

then $H_{r,z}$ is positive definite when $r$ is totally positive. We claim $\text{im} H_{r,z} = E_{r,z}$. It is enough to check this on $\Lambda_z$, and since $E_z$ is alternating, it suffices to check this on pairs of vectors where the first one is in $\mathcal{O}_F \cdot (z_1, z_2)$ and the second is in $\mathcal{O}_F \cdot (1, 1)$:

$$\text{im} H(((i_1(\alpha)z_1, i_2(\alpha)z_2), (i_1(\beta), i_2(\beta)))) = \text{im} \frac{i_1(\alpha)z_1 i_1(\beta) i_1(r)}{\text{im}(z_1)} + \ldots$$

$$= i_1(\alpha)i_1(\beta) i_1(r) + \ldots$$

$$= \text{tr}_{F/\mathbb{Q}} \alpha \beta r$$

$$= E((\alpha, 0), (0, \beta))$$

$$= E_z(\alpha \cdot (z_1, z_2), \beta \cdot (1, 1))$$

We conclude that $A_z$ is an abelian variety, since (for $r$ totally positive in $D_{F/\mathbb{Q}}^{-1}$) it is polarized by $H_{r,z}$. The form $H_{r,z}$ corresponds to a polarization $\lambda_{r,z} : A_z \rightarrow A_z^\vee$ of degree $N(rD_{F/\mathbb{Q}})^2$. If we choose a particular $r$ with $(r) = D_{F/\mathbb{Q}}^{-1}$, then $\lambda_{r,z}$ is principal. The abelian variety $A_z$ comes equipped with an action of $\mathcal{O}_F$, and the polarization is $\lambda_{r,z}$ is $\mathcal{O}_F$-linear.

**Theorem 1.2.1.** $\Gamma \setminus X$ is in bijection with the set of triples $(A, \iota, \lambda)$, where $A/\mathbb{C}$ is an abelian variety, $\iota : \mathcal{O}_F \rightarrow \text{End} A$ is a homomorphism, and $\lambda : A \rightarrow A^\vee$ is an $\mathcal{O}_F$-linear principal polarization.
To compactify \( Y = \Gamma \backslash X \), we need to add a cusp for each \( \Gamma \)-orbit of rational parabolic subgroups of \( G/\mathbb{Q} \). As with the modular curve case, the set of such subgroups is a projective space, \( \mathbb{P}^1(F) \). Thus the cusps of \( Y \) are in bijection with \( \Gamma \)-orbits on \( \mathbb{P}^1(F) \). In fact the quotient set is the class group of \( F \). Proof: given a ratio \([\alpha : \beta]\) in \( \mathbb{P}^1(F) \), associate the fractional ideal \( \alpha \mathcal{O}_F + \beta \mathcal{O}_F \). This is well-defined because if you scale \( \alpha \) and \( \beta \) by the same \( \gamma \in \mathbb{F}^\times \), then the ideal changes by a principal ideal, and clearly translating by \( \text{SL}_2(\mathcal{O}_F) \) doesn’t change anything. It is surjective because every ideal is generated by two elements.

For injectivity: say
\[
\alpha \mathcal{O}_F + \beta \mathcal{O}_F = \alpha' \mathcal{O}_F + \beta' \mathcal{O}_F
\]
in the class group of \( F \). After replacing \( \alpha, \beta \) by a scalar multiple, we may assume that this is an equality of ideals. Let \( I^{-1} \) be the common ideal, so that
\[
\mathcal{O}_F = \alpha I + \beta I = \alpha' I + \beta' I.
\]
Let \( a, b, a', b' \) be such that
\[
1 = \alpha a + \beta b = \alpha'a' + \beta'b'.
\]
Let
\[
M = \begin{pmatrix} \alpha & -b \\ \beta & a \end{pmatrix}, \quad N = \begin{pmatrix} \alpha' & -b' \\ \beta' & a' \end{pmatrix},
\]
so that \( \det M = \det N = 1 \). We have
\[
M \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \text{and} \quad N \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix},
\]
so that \( NM^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha' \beta' \).

Let \( \overline{Y} \) be the disjoint union of \( Y \) and \( \text{SL}_2(\mathcal{O}_F) \backslash \mathbb{P}^1(F) \). Turn \( \overline{Y} \) into a topological space by declaring a system of neighborhoods of \((i \infty, i \infty)\) to be
\[
\left\{ (z_1, z_2) \in \mathcal{H} \times \mathcal{H} \mid \text{im } z_i > r \right\}, \quad r \in \mathbb{R}_{>0}
\]
Acting on these by elements of \( \text{SL}_2(F) \), we get neighborhoods for all the cusps. At this point one gives \( \overline{Y} \) the structure of a complex analytic space (but it is quite complicated). It turns out that unlike the case of modular curves, the cusps are actually quite singular.

Hirzebruch constructed a desingularization of \( \overline{Y} \), whose geometry involves the continued fraction expansion of \( \sqrt{d} \). Using this he was able to prove:
Theorem 1.2.2. Let $p > 3$ be a prime with $p \equiv 3 \pmod{4}$. Assume that $h(\mathbb{Q}(\sqrt{p})) = 1$. Write $\sqrt{p} = [a_0, a_1, \ldots, a_s]$ with $s$ minimal. Then

$$h(\mathbb{Q}(\sqrt{-p})) = \frac{1}{3} \sum_{j=1}^{s} (-1)^j a_j.$$