FREDHOLM ALTERNATIVE
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(I) Statement and Theory of Fredholm Alternative

Let \( \mathcal{X}, \mathcal{Y} \) be Banach spaces, \( A \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \), \( y \in \mathcal{Y} \). We want to know when the equation \( Ax = y \) is solvable for \( x \in \mathcal{X} \).

(a) Intuition

Let \( T \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \), closed. Then \( T^* \in \mathcal{L}(\mathcal{Y}^*, \mathcal{X}^*) \) and \( N(T^*) \perp R(T) \) when interpreted as the \( \mathcal{Y} - \mathcal{Y}^* \) inner product. We wish to know for which values of \( b \in \mathcal{Y} \) there exists a solution to \( Tx = b \); in other words, we wish to know which values of \( b \) are in the range of \( T \). Let \( y^* \in N(T^*) \). Then, \( b \in R(T) \iff \langle b, y^* \rangle = 0 \).

(b) Fredholm Alternative

Let Let \( \mathcal{X}, \mathcal{Y} \) be Banach spaces, \( A \) be a Fredholm Operator, \( b \in \mathcal{Y} \). Then \( Ax \) solvable for \( x \) \( \iff B \in R(A) \iff \langle b, y^* \rangle \forall y^* \in N(A^*) \). [2]

Def: We say \( A \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \) is a Fredholm operator if \( R(A) \) is closed and if both \( \dim N(A) < \infty \) and \( \text{codim} R(A) < \infty \). Then we define the Fredholm index \( \text{ind} A = \dim N(A) - \text{codim} R(A) \). [3]

ex: Let \( T : \mathbb{R}^n \to \mathbb{R}^m \) for \( n > m \). Then let \( \dim R(T) = l, l \leq m \) so that \( \text{codim} R(T) = m - l \). Then \( N(T) = n - l \) and \( \text{ind} T = n - m \). Thus we can interpret the Fredholm index as the ”number of variables minus the number of equations.”

(II) Counter-example if \( T \) is Not Closed [4]

Consider the operator \( T : L^2(\mathbb{R}, \mathbb{R}) \to L^2((R), (R), u \to u_x, \) with domain \( H^1(\mathbb{R}, \mathbb{R}) \). Then the \( L^2 \) adjoint is \( T^*u = -u_x \) by integration by parts, defined again on \( L^2 \).

First calculate \( N(T^*) = \{ v \in L^2 : -u_x = 0 \} \), which is clearly given by the \( \text{span}\{1\} \).

However, \( 1 \notin L^2(\mathbb{R}) \), hence \( N(T^*) = \{0\} \). Then, if the Fredholm Alternative applies, we would have \( R(T) = L^2 \). We construct a counter example to show that this is not true.

Take any odd, continuous, bounded function \( h(x) \) with
\[ h(x) = \frac{1}{|x|^{\frac{3}{2}}} \quad |x| \geq 1 \]

Then \[ \int_{\mathbb{R}} h(x)^2 dx = \int_{-1}^{1} h(x)^2 dx + 2 \int_{1}^{\infty} \frac{dx}{|x|^3} \leq 2 \left[ \sup(h(x)) \frac{x}{x^2} \right] = 2[\sup(h(x) + 2), and thus \( h(x) \in L^2 \). However, we will show that \( h(x) \notin R(T) \). If it were, then \( Tu = h \) would be solved by \( u_x = h(x) \) so that \( u(x) \) is given by

\[ u(x) = a + \int_{1}^{x} h(y) dy \]

For \( x \geq 1 \) we have \( u(x) = a - \frac{2}{\sqrt{|x|}} = a + 2 - \frac{2}{\sqrt{x}} \) so that \( \forall a \in \mathbb{R}, u(x) \notin L^2 \).

Therefore, \( h \notin R(T) \).

So what went wrong with the hypotheses of the Fredholm Alternative? We do not have \( R(T) \) closed in \( L^2 \) since \( h(x) \) can be approximated by the \( L^2 \) functions

\[ h_n(x) = \begin{cases} \frac{1}{|x|^{\frac{3}{2}} + \pi} & |x| \geq 1 \\ \frac{1}{h(x)} & |x| \leq 1 \end{cases} \]

(III) **Application to Wave Trains** [1]

(a) Review of Setup

Recall that we have been studying wave train solutions \( u(x,t) = u_0(\omega_0 t - k_0 x) \) to

\[ \partial_t u = D \partial_{xx} u + f(u) \]

where \( u_0(\theta) \) is 2\( \pi \)-periodic in its argument. Substituting \( u_0 \) into (4) we see that \( u_0 \) must satisfy

\[ k^2 D \partial_{\theta\theta} u - \omega \partial_{\theta} u + f(u) = 0 \]

for \( k = k_0 \) and \( \omega = \omega_0 \), with linearization

\[ L_0 = k^2 D \partial_{\theta\theta} - \omega \partial_{\theta} + f'(u_0(\theta)) \]

We assumed that \( \lambda = 0 \) is a simple eigenvalue of \( L_0 \) so that
• its null space is one dimensional, spanned by the derivative $\partial_\theta u_0$ of the wave train

• $\lambda = 0$ simple also implies that $\partial_\theta u_0$ is not in the range of $L_0$. If it were, then $\exists u_1$ s.t. $L_0u_1 = u_0$, implying that $\lambda = 0$ is part of a Jordan block of dimension at least 2, which violates the simple hypothesis.

• We have shown that there exists a neighborhood $V$ of $(k_0, \omega_0)$ such that wave trains given by $u(x,t) = u_0(\omega_{nl}(k)t - kx; k)$ are still solutions to (4). The purpose of this lecture is to derive an expansion for $\omega_{nl}$ near $k_0$.

(b) Calculation

We assume that the hypotheses of the Fredholm Alternative hold; this will be verified later. The adjoint of $L_0$ is

$$L_0^\dagger = k_0^2 D\partial_{\theta\theta}u + \omega_0 \partial_\theta u + f'(u_0(\theta))^T u$$

(7)

The null space of $L_0^\dagger$ is spanned by $u_{ad}$, normalized so that $\langle u_{ad}, \partial_\theta u_0 \rangle_{L^2} = 1$. Note that this normalization is possible by the Fredholm Alternative since $\partial_\theta u_0$ is not in the range of $L_0$.

We then substitute $u = u_0$ and $\omega = \omega_{nl}(k)$ into (5) and take the derivative with respect to $k$

$$(k^2 D\partial_{\theta\theta} - \omega \partial_\theta + f'(u_0(\theta)))\partial_k u + 2k D\partial_{\theta\theta}u_0 - \frac{dw_{nl}(k)}{dk} \partial_\theta u_0 = 0$$

(8)

Evaluated at $u = u_0$ and $k = k_0$ we have the first two derivatives of (5) are given by

$$L_0\partial_k u_0 + 2k_0 D\partial_{\theta\theta}u_0 - \frac{dw_{nl}(k_0)}{dk} \partial_\theta u_0 = 0$$

$$L_0\partial_{kk} u_0 + 2k_0 D\partial_{\theta\theta k}u_0 - \frac{dw_{nl}(k_0)}{dk} \partial_{k\theta} u_0 + f''(u_0)[\partial_k u_0(\theta), \partial_\theta u_0(\theta)] + 2D\partial_{\theta\theta}u_0$$

$$+ 2k_0 D\partial_{\theta\theta k}u_0 - \frac{d^2w_{nl}(k_0)}{dk^2} \partial_{kk} u_0 - \frac{d^2w_{nl}(k_0)}{dk^2} \partial_\theta u_0 = 0$$

(9)

Rearranging we find
\[ \mathcal{L}_0 \partial_k u_0 = -2k_0 D \partial_\theta u_0 + \frac{d w_{nl}(k_0)}{d k} \partial_\theta u_0 \]

(10)

\[ \mathcal{L}_0 \partial_{kk} u_0 = -4k_0 D \partial_\theta \partial_k u_0 + \frac{2 d w_{nl}(k_0)}{d k} \partial_{kk} u_0 - f''(u_0) [\partial_k u_0(\theta), \partial_k u_0(\theta)] - 2D \partial_\theta \partial_k u_0 + \frac{d^2 w_{nl}(k_0)}{d k^2} \partial_\theta u_0 \]

We can conclude from (10) that the right hand side of each equation must be in the range of \( \mathcal{L}_0 \). We apply the Fredholm Alternative, which states that therefore \( \langle \text{RHS}, u_{ad} \rangle = 0 \). Using the normalization property we have

\[ 0 = \langle -2k_0 D \partial_\theta u_0 + \frac{d w_{nl}(k_0)}{d k} \partial_\theta u_0, u_{ad} \rangle \]

\[ = \langle -2k_0 D \partial_\theta u_0, u_{ad} \rangle + \frac{d w_{nl}(k_0)}{d k} \]

(11)

\[ 0 = \langle -4k_0 D \partial_\theta \partial_k u_0 + \frac{2 d w_{nl}(k_0)}{d k} \partial_{kk} u_0 - f''(u_0) [\partial_k u_0(\theta), \partial_k u_0(\theta)] - 2D \partial_\theta \partial_k u_0, u_{ad} \rangle + \frac{d^2 w_{nl}(k_0)}{d k^2} \partial_\theta u_0 \]

From which we can solve for the first two terms in the expansion of \( w_{nl}(k_0) \).

\[ \frac{d w_{nl}(k_0)}{d k} = c_g = \langle 2k_0 D \partial_\theta u_0, u_{ad} \rangle \]

(12)

\[ \frac{d^2 w_{nl}(k_0)}{d k^2} = \langle -4k_0 D \partial_{\theta \theta} u_0 + 2c_g \partial_{\theta k} u_0 - f''(u_0) [\partial_k u_0(\theta), \partial_k u_0(\theta)] - 2D \partial_{\theta \theta} u_0, u_{ad} \rangle \]

(c) Expansion of \( \lambda_{lin} \)

We can similarly solve for the first two terms in the expansion of \( \lambda_{lin}(0) \) as follows. Recall the set-up for \( \lambda_{lin} \). We linearize (4) in the comoving frame \( \theta = \omega_0 t - k_0 x \) to find

\[ \partial_t u = k_0^2 D \partial_\theta \theta u - \omega_0 \partial_\theta u + f'(u_0(\theta)) u \]

and substitute in the ansatz \( u(\theta) = e^{-\nu \theta / k_0} v(\theta; \nu) \) to obtain

\[ \mathcal{L}_\nu v = k_0^2 D \left( \partial_\theta - \frac{\nu}{k_0} \right)^2 v - \omega_0 \left( \partial_\theta - \frac{\nu}{k_0} \right) v + f'(u_0(\theta)) v \]

(14)
Since \( \lambda = 0 \) is a simple eigenvalue, we find that there is an analytic curve of eigenvalues \( \lambda_{lin}(\nu) \) near zero for \( \nu \in i\mathbb{R} \) near zero. Therefore, we have the following equation

\[
k_0^2 D \left( \partial_\theta - \frac{\nu}{k_0} \right)^2 v - \omega_0 \left( \partial_\theta - \frac{\nu}{k_0} \right) v + f'(u_0(\theta))v = \lambda_{lin}(\nu)v
\]

We proceed as before taking the first and second derivatives of (15) with respect to \( \nu \) evaluated at \( \nu = 0 \), using the fact that \( v = \partial_\theta u_0 \) at \( \nu = 0 \). We find

\[
L_0 \partial_\nu v = (\lambda'_{lin}(0) - c_p) \partial_\theta u_0 + 2k_0 D \partial_\theta \partial_\nu u_0
\]

\[
L_0 \partial_\nu v = \lambda''_{lin}(0) \partial_\theta u_0 + 2\lambda'_lin(0) \partial_\nu v + 4k_0 D \partial_\theta \partial_\nu v - 2c_p \partial_\nu v - 2D \partial_\theta u_0
\]

where \( c_p = \frac{u_0}{k_0} \) is the phase speed at \( k_0 \). Using the Fredholm Alternative as above we get

\[
\lambda'_{lin}(0) = c_p - c_g
\]

\[
\lambda''_{lin}(0) = \langle 4k_0 D \partial_\theta \partial u_0 + 2D \partial_\theta u_0 \rangle
\]

(d) Verification of the Hypotheses of the Fredholm Alternative

We now verify that the hypotheses of the Fredholm Alternative have indeed been met.

(i) \( L_0 \) is closed

By writing \( L_0 \) as a system of first order ODEs, we can reduce the problem to showing that \( T = \frac{\partial}{\partial x} - A \) is a closed operator from \( L^2_{\text{per}}(0, 2\pi) \to L^2_{\text{per}}(0, 2\pi) \) with domain \( H^1_{\text{per}}(0, 2\pi) \).

We take a sequence \( \{h_n\} \in L^2_{\text{per}}(0, 2\pi) \) such that \( h_n \xrightarrow{n \to \infty} h \in L^2_{\text{per}}(0, 2\pi) \) and such that \( \forall n \exists u_n : Tu_n = h_n \). We need to show that \( u_n \to u \in H^1_{\text{per}}(0, 2\pi) \) and \( Tu = h \).

Let \( \Phi(x, 0) \) be the fundamental matrix solution to \( Tu = 0 \). Then we have

\[
u_n(x) = \Phi(x, 0)u_n(0) + \int_0^x \Phi(x, y)h_n(y)dy\]
We know that the integral converges because \[ \int_0^\pi \Phi(x,y)h_n(y)dy = \langle \Phi(x,y), h_n(y) \rangle_{L^2}, \]
so by Hilbert space theory we can conclude that \( \langle \Phi(x,y), h_n(y) \rangle_{L^2} \rightarrow \langle \Phi(x,y), h(y) \rangle_{L^2} \). Therefore, it suffices to show that \( u_n(0) \rightarrow u(0) \). We use the fact that \( u_n(2\pi) = u_n(0) \forall n \) to get

(19) \[ u_n(0) = u_n(2\pi) = \Phi(2\pi,0)u_n(0) + \int_0^{2\pi} \Phi(x,y)h_n(y)dy \]

and it suffices to solve

(20) \[ (I - \Phi(2\pi,0))u_n(0) = \int_0^{2\pi} \Phi(x,y)h_n(y)dy \]

We decompose \( \mathbb{R}^n \cong N(I - \Phi(2\pi,0)) + X \) so that we can write \( u_n(0) = \alpha_n u_n^* + a_n \) where \( u_n^* \in N(I - \Phi(2\pi,0)) \) and \( a_n \in X \). We can redefine \( \alpha_n \) so that they converge since they are in the null space. Then we have

(21) \[ (I - \Phi(2\pi,0))a_n(0) = \int_0^{2\pi} \Phi(x,y)h(y)dy \]

which we can invert on the range of \( I - \Phi(2\pi,0) \) to get

(22) \[ a_n(0) = (I - \Phi(2\pi,0))^{-1} \int_0^{2\pi} \Phi(x,y)h(y)dy \]

The integral on the right hand side converges as noted above. It is in the range of \( I - \Phi(2\pi,0) \) since the range is a finite dimensional subspace of \( \mathbb{R}^n \) and necessarily closed; therefore, the right hand side converges, and since \( X \) is again a finite dimensional subspace \( a_n(0) \rightarrow a(0) \in X \).

Now we can write \( u_0(x) = \alpha u_n^* + a \), and \( u(x) = \Phi(x,0)u(0) + \int_0^{2\pi} \Phi(x,y)h(y)dy \)

necessarily satisfies \( Tu = h \). Moreover, \( u(x) \in H^1_{per}(0,2\pi) \) since \( \Phi(x,y) \) is.

(ii) \( \dim N(\mathcal{L}_0) < \infty \)

By hypothesis that \( \lambda = 0 \) is simple.

(iii) \( \text{codim } R(\mathcal{L}_0) < \infty \)

As we saw in bullet (i), \( \text{ind}(\mathcal{L}_0) \) is equivalent to \( \text{ind}(I - \Phi(2\pi,0)) \). We show that \( \text{ind}(I - \Phi(2\pi,0)) = 0 \), hence \( \text{codim } R(\mathcal{L}_0) = 1 < \infty \). The
claim follows from the fact that $I - \Phi(2\pi, 0) : \mathbb{R}^n \to \mathbb{R}^n$, and hence as we showed previously, $\text{ind}(I - \Phi(2\pi, 0)) = n - n = 0$.

(e) Observation about Fredholm Alternative

We make an observation of the formulation of $L_0u = h$ in terms of

$$(I - \Phi(2\pi, 0)) u(0) = \int_0^{2\pi} \Phi(x, y) h(y) dy.$$  

(23)

We make the assumption that $N(I - \Phi(2\pi, 0))$ is 1 dimensional so that $\exists! \Psi_0 \perp R(I - \Phi(2\pi, 0))$. Then (23) has a solution $\iff$

$$0 = \int_0^{2\pi} \langle \Psi_0, \Phi(2\pi, y) h(y) \rangle dy$$

$$= \int_0^{2\pi} \langle \Phi(2\pi, y)^\dagger \Psi_0, h(y) \rangle dy$$

$$= \int_0^{2\pi} \langle \Psi(x), h(y) \rangle dy$$  

(24)

where $\Psi(x)$ solves the adjoint problem $\frac{dw}{dx} = -A^\dagger(x)w$.

References


