On the Distance Between Network Links and its Relation to Covariance in Network Tomography

David Chua, Eric D. Kolaczyk, Mark Crovella, and Anukool Lakhina

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Abstract

There is a need in the analysis of computer network traffic data for metrics useful in characterizing the distance between network links, particularly in connection with understanding patterns in the covariance of measurements across links. We offer a path-based notion of distance and provide evidence that, contrary to a more abstract notion of link distance introduced recently, covariance in a simplified network tomography model decays linearly with respect to this distance on a log-log scale. Implications of this finding are discussed.

Index Terms - Graphs, metrics, networks.

1 Introduction.

In the field of computer network traffic analysis, the importance of studying traffic simultaneously on multiple network links is coming to be increasingly recognized as researchers begin to tackle high-level, whole-network tasks, such as the network-wide detection of anomalies and attacks. As a fundamental tool, therefore, it is important to have metrics useful for characterizing the distance between network links, particularly in connection with understanding patterns

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in the covariance of measurements across links. The lack of such metrics recently was noted, and a proposal was made for using the $\ell_1$ distance between rows of the underlying traffic routing matrix [14].

Here we offer a path-based notion of distance between network links, related to the notion of 'hop distance' in networking. We then provide results of an empirical study characterizing the relation between covariance and this distance, in a simplified version of the network tomography model for which covariance between links reduces to the number of common traffic flows. It is found that the average of the covariance studied (where the average is across all links a fixed distance apart) decays in a roughly linear fashion with respect to our distance metric on a log-log scale. The same quantity compared to the more abstract metric in [14] does not exhibit this behavior. We complement these findings with a small simulation study, involving the complete graph on $N$ nodes and random routing weights, for which the results are found to be similar, suggesting that the observed pattern is primarily a function of shortest-path routing. We close with additional discussion on the implications of the findings herein.

2 Distance Between Network Links

2.1 Background

Let $G = (V, E)$ be a strongly connected, directed network graph, where $V$ is the set of vertices (or nodes) and $E$ is the set of all pairs of vertices $e = (u, v)$, for $u, v \in V$, for which there exists a directed edge (or link) from $u$ to $v$. The traffic on this network will be assumed to consist of flows between origin-destination (OD) pairs $(a, b)$, for $a, b \in V$. Routing between OD pairs will be assumed to be static, and so the routes taken by flows can be represented as an $m \times n$ binary matrix $A = (a_{i,j})$, with the $m$ rows corresponding to all links in $G$, the $n$ columns corresponding to all routed OD pairs, and $a_{i,j} = 1$ if and only if the flow between the $j$-th OD pair traverses the $i$-th link. We will assume the routing matrix $A$ to be known.

Coates et al. [2] have noted that the framework defined above arises in many contexts involving data networks. Often the measurements taken on such a network will be analyzed in
a link-indexed fashion. Therefore, as a fundamental quantity necessary for pursuing any sort of simultaneous analysis of measurements across links, good metrics describing the distance between links are important. Citing a general lack of such metrics, Vardi [14] introduced the quantity

\[ d_V(e_1, e_2) = N(e_1) + N(e_2) - 2N(e_1, e_2) \]  

(1)

for characterizing the distance between the two links \( e_1 \) and \( e_2 \), where \( N(e_1, e_2) \) is the number of flows passing over both links \( e_1 \) and \( e_2 \) and \( N(e_i) = N(e_i, e_i) \) is the number of flows over \( e_i \), \( i = 1, 2 \). That is, \( d_V \) counts the total number of flows passing over one or the other but not both of the links. Because \( d_V \) is equivalent to the \( \ell_1 \) distance (and also the squared Euclidean and Hamming distances) between the rows \( A_{e_1} \) and \( A_{e_2} \) of the routing matrix corresponding to \( e_1 \) and \( e_2 \), as vectors in \( \mathbb{R}^n \), it follows that it is a metric in a formal sense, satisfying the necessary conditions of non-negativity, symmetry, and the triangle inequality.

2.2 A Path-Based Notion of Distance

From an operational perspective, the derivation of a path-based notion of distance is attractive, with its definition inspired by consideration of the physical flow of traffic over the network. Recall that a path \( \mathcal{P} \) between two vertices \( u, v \in V \) on a directed graph is a sequence of alternating vertices and edges in which each edge begins at the previous vertex and ends at the subsequent vertex and none of the vertices or edges occur more than once. Extending this concept slightly, we will define a path \( \mathcal{P}_{e_1 \rightarrow e_2} \) from an edge \( e_1 = (a_1, b_1) \) to another edge \( e_2 = (a_2, b_2) \) to be a path \( \mathcal{P} \) between \( a_1 \) and \( b_2 \) that first traverses the edge \( e_1 \) and last traverses the edge \( e_2 \). Now let \( \Pi(G) \) be the set of all paths on \( G \) and, for an arbitrary path \( \mathcal{P} \in \Pi(G) \), let \( H(\mathcal{P}) \) be the number of edges in \( \mathcal{P} \) i.e. the so-called ‘hop length’ of \( \mathcal{P} \). Additionally, let \( \Pi(G, A) \subseteq \Pi(G) \) be those paths that are actually routed under \( A \), as it is only these latter that are relevant to the flow of traffic. We adopt the convention that each routed path \( \mathcal{P} \) between any two origin-destination nodes is unique, that is, that all traffic from the origin to the des-
tion flows over the same set of links. This is consistent with the vast majority of routing configurations in both local area and wide area data networks.

We proposed the following path-based distances for edges on a data network.

**Definition 2.1** Let $e_1, e_2 \in E$. Define

$$d_{H|A}(e_1, e_2) = \begin{cases} H(\mathcal{P}_{e_1 \rightarrow e_2}) - 1, & \text{if } \exists \mathcal{P}_{e_1 \rightarrow e_2} \in \Pi(G, A) \\ \infty, & \text{otherwise} \end{cases}$$

(2)

to be the routed distance from $e_1$ to $e_2$, and

$$d_{H|A}^{\text{sym}}(e_1, e_2) = \min\{ d_{H|A}(e_1, e_2), d_{H|A}(e_2, e_1) \}$$

(3)

to be the routed distance between $e_1$ and $e_2$.

A few remarks are in order regarding the above definitions.

**Remark 1:** The use of two cases in defining $d_{H|A}$ in (2) follows from the need to distinguish between routed and non-routed paths. Note that this distinction is relevant to paths between arbitrary links, as opposed to paths between arbitrary nodes, because, while we assume that for all nodes there is a unique path routed between them under $A$, this will not necessarily be the case when dictating that a particular pair of links be the first and last traversed by such a path. In other words, there may not be a path $\mathcal{P}_{e_1 \rightarrow e_2}$, for $e_1 = (a_1, b_1)$ and $e_2 = (a_2, b_2)$, from $a_1$ to $b_2$ that is actually routed. In this case, since there is no common traffic passing over both links, we adopt the convention that the two links are infinitely far apart.

**Remark 2:** Note that $d_{H|A}$ is an asymmetric notion of distance since, given that $G$ is directed, a path $\mathcal{P}_{e_1 \rightarrow e_2}$ that starts with an edge $e_1$ and ends with an edge $e_2$ cannot simply be reversed to obtain a path $\mathcal{P}_{e_2 \rightarrow e_1}$.

**Remark 3:** Formally, the distance $d_{H|A}$ is a quasi-semi-metric, failing to satisfy both symmetry and the triangle inequality (but satisfying non-negativity). That
Figure 1: Counter-example illustrating the failure of the distance $d_{H|A}$ to satisfy the triangle inequality, where each link is shown accompanied by a routing weight and routing between any two nodes is assumed to follow that path for whom the sum of the routing weights is minimized.

the triangle inequality can fail may be shown by simple counter-example, such as in Figure 1. Here the failure derives from the fact that while $d_{H|A}$ is given in units of hops, the underlying routing protocol typically is based on ‘shortest-path’ comparison of routing weights (whose units typically are not reflective of any natural physical quantity) or similar. Similarly, $d_{H|A}^\text{sym}$ is a semi-metric, given that it is a symmetrized version of $d_{H|A}$. The decision to use the minimum function to induce symmetry is due to the fact that empirical experience shows that a large proportion of the time one or both of the quantities $d_{H|A}(e_1, e_2)$ and $d_{H|A}(e_2, e_1)$ are infinite.

Remark 4: Computation of $d_{H|A}(e_1, e_2)$, for a given pair of links $e_1 = (a_1, b_1)$ and $e_2 = (a_2, b_2)$, is straightforward. The column of $A$ corresponding to the flow $a_1 \rightarrow b_2$ is extracted. If the entries of this column corresponding to $e_1$ and $e_2$ are both one, then $P_{e_1 \rightarrow e_2} \in \Pi(G, A)$. In that case, the required distance is simply the sum of this column; otherwise, it is set to infinity.

3 Covariance versus Link Distance in Network Tomography

3.1 Network Tomography Model

The basic network tomography model (e.g., [13]) specifies that, for a given unit time period (e.g., 10 ms, 1 sec, etc.), the underlying flow data $\mathbf{x} = (x_1, \ldots, x_n)'$ between the $n$ OD pairs and the measurements $\mathbf{y} = (y_1, \ldots, y_m)'$ observed on the $m$ links are related through the linear equation $\mathbf{y} = A\mathbf{x}$. Generally interest is in predicting the unknown flows $\mathbf{x}$, as a function of time,
from the observed data $y$, for which many methods have now been proposed. The difficulty of the problem derives from its ill-posedness, given that typically $n \gg m$.

Recently Liang and Yu [7] suggested a method based on a pseudo-likelihood approach, in which information from the link measurements is incorporated in a pair-wise fashion, across all pairs. Vardi [14] suggests that the efficiency of such an estimator might be improved through incorporation of weights proportional to the distance between pairs of links, presumably emphasizing pairs with smaller covariance. As a small step in this direction, we explore in this section the relationship between the covariance of the link measurements $y$ and the distance $d$ between the links. Denoting $\Lambda = \text{cov}(x)$, the covariance of the link measurements is given by $\Sigma = \text{cov}(y) = \Lambda \Lambda'$. It is common to assume that the flow covariance matrix $\Lambda$ is diagonal i.e., $\Lambda \propto \text{diag}(\lambda)$, an assumption that can result, for example, from beginning with a model in which flows are assumed to follow independent Poisson distributions with intensities $\lambda$. In this case the elements $\Sigma_{i,j}$ are simply the total flow intensity common to links $e_i$ and $e_j$.

Here we will make the further simplifying assumption that $\Lambda \propto I_{n \times n}$, in which case $\Sigma \propto AA'$ and therefore $\Sigma_{i,j} \propto N(e_i, e_j)$. That is, we assume a (roughly) common level of flow intensity across OD pairs, which reduces the covariance between links to a value proportional to the number of common flows. This simplification allows us to isolate the effects of the underlying network topology on the covariance and its relation to distance, without the additional complication of flow intensities that are unknown and varying in time. Perhaps surprisingly, however, preliminary examination of actual IP-flow data has shown our assumption to be quite reasonable in certain contexts, such as that of the Abilene network (an Internet2 high-performance backbone network). We comment further on this issue, and the implication of our results on the more general case, in section 4.

3.2 Covariance as a Function of Link Distance

3.2.1 Illustration: Rocketfuel Topologies

Our goal is to study the relationship of $\Sigma_{i,j}$ as a function of $d(e_i, e_j)$, for the distances $d_V$ and $d_{H/A}^{\text{sym}}$. The simplification described above effectively means using $N(e_i, e_j)$ as a proxy
Figure 2: Log-log plot of the proxy covariance $N(e_i, e_j)$ as a function of $d_{H[A]}^{sym}(e_i, e_j)$. Boxplots summarize distribution of covariances at each distance. Also shown (filled circles) are the logarithms of the average covariances at each distance.

to $\Sigma_{e_i,e_j}$, yet even this information typically is unknown (or, more precisely, unavailable) to most researchers for networks of any real magnitude. Recently, however, through the so-called ‘Rocketfuel’ project, the underlying topology and routing information has been independently inferred for the networks of six internet service providers (ISPs), and verified as a “good” to “excellent” match by a number of these providers [8]. The Rocketfuel data are for six ISPs (Telstra, Sprint, Fibone, Tiscali, Exodus and AboveNet) with networks in Australia, the U.S. and Europe. Sizes of the inferred networks range from 79 routers and 147 links for Exodus to 315 routers and 972 links for Sprint.

Using the Rocketfuel data, for each of the six networks we computed $N(e_i, e_j), d_{H[A]}^{sym}(e_i, e_j)$, and $d_Y(e_i, e_j)$. It was found that the most distinct relationship between this proxy covariance and these distances manifested itself on a log-log scale. For the path-based distance $d_{H[A]}^{sym}$, shown in Figure 2, the covariances line up over one of only a relatively few possible distances, due to our use of ‘hops’ as units. (Recall that the diameter of classical random graphs, for example, scales only like $\log_2(N)$, where $N$ is the number of nodes [1].) At each distance, the log-covariances seem to follow a distribution highly skewed to the right. In addition, there seems to be some indication that these distributions scale with distance. Even stronger, however, is the evidence
Figure 3: Log-log plot of $N(e_i, e_j)$ as a function of $d_V(e_i, e_j)$. Individual points denote averages over bins of distance width 1. Contour plots are super-imposed over points to better indicate relative concentration of points.

of a fairly linear decay in the logarithm of the average covariance within each distance, as a function of log-distance, which is shown over-plotted on the boxplots in the figure.

In contrast to these results, plots of covariance against the distance $d_V$, (again on a log-log scale), shown in Figure 3, reveal a much more diffuse behavior and one whose tendency is to increase with distance. The diffuseness of the points is due partly to the fact that $d_V$ is not constrained to units of hops, but rather is bounded above by the the total number of flows $n$.

Also, it should be noted that whereas the majority (between 77 and 94 percent) of edge-pairs were judged infinitely far apart according to the distance $d_{sym}^H|A$, and hence not shown on the plots in Figure 2 (i.e., they are placed at infinity, where their covariances are all identically equal to zero), all edge-pairs have finite distance according to $d_V$ and hence contribute to the plots in Figure 3. That there is an upward trend relating covariance to the distance $d_V$ is not surprising, considering the expression in (1) and the fact that $\Sigma_{i,j} \propto N(e_i, e_j)$ here.

3.2.2 Numerical Experiment: The case of $K_N$

The above results suggest the value of the distance $d_{sym}^H|A$ for characterizing the behavior of the covariance $\Sigma_{i,j}$ in the network tomography model. Furthermore, in analogy to what is often
observed in classical spatial statistics (e.g., [3]), the evidence suggests an interesting decay of covariance as a function of this distance. Given the complexities inherent in the topology of large-scale computer traffic networks such as those studied above, it is interesting to see to what degree the behavior of covariance versus distance may be replicated through a simple, artificial network model. For this we consider the undirected, complete graph on $N$ nodes i.e., $G = K_N$, and equip it with a routing matrix $A$ by generating independent and identically distributed $\text{Exp}(1)$ weights for each edge and computing the shortest path between each pair of nodes. Despite its simplicity, this model has been found to produce networks with some characteristics surprisingly similar to those observed in the internet (e.g., [9, 12]).

The covariance in this context will be assumed to take the form $\Sigma = AA^T$ identically, and therefore $\Sigma_{i,j} = N(e_i, e_j)$ for each pair of edges $e_i$ and $e_j$. Motivated by the strong linear trend exhibited in the plots of Figure 2, we focus our attention here on the behavior of the quantity

$$\text{AveCov}(d) = \frac{\sum_{(i,j):d(e_i,e_j)=d}N(e_i,e_j)}{\#\{(i,j):d(e_i,e_j)=d\}},$$

as a function of distance $d$, over repeated draws of $A$. A total of 300 trials were run, for the case $N = 1000$, the results of which are summarized in Figure 4. Note that a strongly linear behavior, on a log-log scale, again is observed. Given the disparate nature of the Rocketfuel networks and the complete graph $K_N$ with exponential weights, the presence of linear trends in both Figure 2 and Figure 4 suggests that this behavior is due primarily to the use of shortest-path-like routing.

Of independent interest is the observation that the average covariances in our simulation appear to be fairly tightly concentrated within each distance $d^\text{sym}_{H,A} = d$. The question then arises as to whether such concentration can be shown in an analytically precise manner and about what quantity the concentration occurs. Note that one can re-express the within-distance average covariance in (4) as

$$\text{AveCov}(d) = \frac{\sum_{i,j}(L_{i,j} - d)I_{\{L_{i,j}>d\}}}{\sum_{i,j}I_{\{L_{i,j}=d+1\}}}.$$
Figure 4: Boxplots of AveCov(d) (equation (4)) versus d, on a log-log scale, based on 300 trials for the graph $K_{1000}$. Over-plotted on each boxplot (shaded circle) is the log of the ratio in (6). The close match between simulated and analytic values, combined with the limited extent of the boxplots themselves, suggests that the averages in the numerator and denominator of (4), re-expressed in (5), concentrate about their expected values.

where $L_{i,j}$ is the length, in units of hops, of the shortest path between nodes $i$ and $j$. Hence the numerator and denominator in AveCov(d) may be thought of as averages of simple functions of path-lengths $L_{i,j}$, over all $N(N - 1)/2$ OD pairs $(i, j)$. And if the dependence among the $L_{i,j}$ is sufficiently weak, these averages might reasonably be expected to concentrate about their expected values.

An analytic expression for these expected values may be calculated by using the fact that, due to the memoryless property of the exponential distribution, the set of all shortest paths that start with a single given node form a uniform recursive tree (URT). A URT of order $N$ is a tree with $N$ nodes that has been chosen uniformly from the set of $(N - 1)!$ possible recursive trees of order $N$, where a tree $T_N$ is of the latter type if (i) the root is labeled 1 and (ii) the labels along any path from the root to a leaf node form an increasing sequence (e.g., [10]). Viewed from this perspective, $L_{i,j}$ is just the depth of a randomly selected node in a URT rooted at
node $i$, and since the distribution of the latter is known \cite{11}, we can calculate that

$$
\frac{E[(L - d)I_{\{L > d\}}]}{E[I_{\{L = d + 1\}}]} = \frac{\sum_{l=d+1}^{n-1} (l - d) \sum_{k=l+1}^{N} \frac{1}{[k-1]! [k-l]!}}{\sum_{k=d+1}^{N} \frac{1}{(k-1)! [k-l+1]!}},
$$

(6)

where $L$ is the length of a path between a randomly selected pair of nodes $(i, j)$ and $[m]_n$ denotes an unsigned Stirling number of the first kind.

The values in (6) may be calculated easily and are displayed in Figure 4. The close match between the simulated and analytic values suggest (6) as a likely candidate for concentration of the average covariance in (4). The authors currently are pursuing a formal argument along these lines, which would seem to involve quantifying the degree to which two randomly selected paths $\mathcal{P}_{e_{i_1} \rightarrow e_{j_1}}$ and $\mathcal{P}_{e_{i_2} \rightarrow e_{j_2}}$ necessarily share edges. Interestingly, although the asymptotic behavior of a given path length $L_{i,j}$ and the maximal path lengths $\max_j L_{i,j}$ and $\max_{i,j} L_{i,j}$ have been characterized on $K_N$ \cite{5} (see also \cite{4, 12} for related work), the problem of characterizing the portion of two paths common to each other is to the best of our knowledge unsolved.

4 Discussion

Development of useful distance metrics for network data is a non-trivial task. Furthermore, the relation of these distances to such fundamental quantities as covariance, in contexts like network tomography, would seem not only to be an issue of practical importance but also one of potential theoretical interest as well. In closing we note two important points associated with the results presented in this paper.

First, while our empirical results suggest the usefulness of adopting a path-based approach to the notion of network distance, it is clear that one might wish to use physically meaningful units other than ‘hops’. For example, the average round-trip-time (RTT) needed to transmit a packet between an OD pair might better match the needs of particular network applications and could vary in a potentially meaningful fashion with underlying network conditions. However, calculation of this quantity would require information above and beyond that contained in the routing matrix $A$, such as historical traffic data (and possibly ‘live’ traffic data if a highly
adaptive measure were desired).

Second, while our work relating covariance to our distance \( \hat{d}_{H|A}^{\text{sym}} \) involves the simplifying assumption of roughly equal flow intensities \( \lambda \) across all OD-pairs, which is clearly unrealistic, preliminary empirical work with actual IP-flow measurements suggests this can be a reasonable assumption in some contexts. Specifically, using the same IP-flow data described in [6] (consisting of Juniper NetFlow measurements on the Abilene network, at the 5-tuple IP flow level, for five-minute intervals over a one-week period, aggregated by common ingress/egress points at the router level), a simple moving average with a one-hour window was applied to produce estimates of the underlying flow intensity vector \( \lambda^{(t)} \) as a function of time \( t \). These estimates were then used to construct a flow covariance matrix \( \Lambda^{(t)} = \text{diag}(\lambda^{(t)}) \) at each time \( t \), and from those, the link covariance matrices \( \Sigma^{(t)} = \Lambda^{(t)} \Lambda^{(t)}' \). The entries of \( \Lambda \Lambda' \) (fixed over the time intervals of our study) were compared to those of the \( \Sigma^{(t)} \) by computing the coefficient of correlation for the resulting two sets of numbers. These coefficients, surprisingly, were found to range between 0.89 and 0.96, thus providing strong support in the context of the Abilene network for the use of \( \Lambda \Lambda' \) as a proxy for the true covariance \( \Sigma^{(t)} \) across time \( t \).

References


