HIGHER MATHEMATICS
LINEAR ALGEBRA
AND COMPLEX NUMBERS

A textbook with a sample of the test

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Preface

In mathematics and its applications it is often necessary to consider sets of objects, for which so called linear operations are defined: addition and multiplication by a number. The most simple and well known such sets are the set of real numbers and the set of vectors of three-dimensional space. The same operations are defined for the set of matrices of the same dimension, which generalize the notions of number and vector.

Another important generalization of real numbers is the set of complex numbers, without the use of which it is impossible to imagine contemporary mathematics, physics, . . .

This textbook contains the main theory, exercises with detailed solutions and a sample of the test on the themes «Linear algebra» and «Complex numbers».

1. Linear algebra

1.1. Matrices. Operations with matrices

Let us consider a rectangular table of \( m \times n \) numbers

\[
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\]

This table is called a matrix and is denoted in the following way:

\[
\mathcal{A} = \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}.
\]

This matrix has \( m \) rows and \( n \) columns. They say that it is of dimension \( m \times n \).

A matrix that has only one row

\[
\mathcal{A} = \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n}
\end{pmatrix},
\]

is called matrix-row or vector-row.
A matrix, consisting of one column

\[ \mathbf{A} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \]

is called matrix-column or vector-column. A matrix that has an equal number of rows and columns

\[ \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \]

is called a square matrix. A matrix that looks as

\[ \mathbf{D} = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}, \]

is called a diagonal matrix, all the elements of this matrix outside its diagonal being equal to zero. A diagonal matrix, which all diagonal elements being equal to 1:

\[ \mathbf{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \]

is called identity matrix. It is denoted by \( \mathbf{I} \).

**Multiplication of a matrix by a number**

Any matrix can be multiplied by a number. If we multiply a matrix \( \mathbf{A} = (a_{ij}) \) by a number \( \lambda \) then the product is the matrix \( \mathbf{C} = (c_{ij}) \) such that for any \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n \):

\[ c_{ij} = \lambda a_{ij}. \]
Exercise. Find the product of

\[ A = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix} \]

and 3.

Solution:

\[ C = 3 \cdot A = 3 \cdot \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 \cdot 1 & 3 \cdot (-3) \\ 3 \cdot 2 & 3 \cdot 1 \end{pmatrix} = \begin{pmatrix} 3 & -9 \\ 6 & 3 \end{pmatrix}. \]

Addition of matrices

It is possible to add (and subtract) only the matrices that have the same dimension. The sum of two matrices of the same dimension \( A = (a_{ij}) \) and \( B = (b_{ij}) \) is the matrix \( C = (c_{ij}) \) such that

\[ c_{ij} = a_{ij} + b_{ij} \]

for any \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n \).

Exercise. Find the sum and the difference of

\[ A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 4 & 3 \\ -2 & 1 \end{pmatrix}. \]

Solution:

\[ C = A + B = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} + \begin{pmatrix} 4 & 3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 + 4 & 2 + 3 \\ -1 - 2 & 3 + 1 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ -3 & 4 \end{pmatrix}; \]
\[ D = A - B = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} - \begin{pmatrix} 4 & 3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 - 4 & 2 - 3 \\ -1 + 2 & 3 - 1 \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 1 & 2 \end{pmatrix}. \]

Multiplication of matrices

One can multiply matrices only if the number of columns in the left matrix is equal to the number of rows in the right one. The product of two matrices

\[ A = (a_{ij}), \quad i = 1, 2, \ldots, m; \quad B = (b_{jk}), \quad j = 1, 2, \ldots, n; \quad k = 1, 2, \ldots, p; \]

\[ C = A \cdot B = \begin{pmatrix} a_{ij} & b_{jk} \\ \vdots & \vdots \end{pmatrix} \]
is the matrix $C = (c_{ik})$ of dimension $m \times p$ such that

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

for any $i = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$.

Roughly speaking, one multiplies the rows of the left matrix by the columns of the right one. The number of rows in the product is equal to the number of rows in the left multiplier, the number of columns equals to the number of columns in the right one. It is necessary to point out that if we multiply the matrices in the reverse order then the product, in the most cases, will be different.

**Exercise.** Find the product of

$$\mathcal{A} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}.$$

**Solution:**

The product $\mathcal{B} \mathcal{A}$ clearly does not exist.

Let us compute $\mathcal{A} \mathcal{B}$:

$$\mathcal{C} = \mathcal{A} \mathcal{B} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 \cdot 3 + (-1) \cdot 4 & 1 \cdot 1 + (-1) \cdot 2 \\ 2 \cdot 3 + 3 \cdot 4 & 2 \cdot 1 + 3 \cdot 2 \\ 1 \cdot 3 + 2 \cdot 4 & 1 \cdot 1 + 2 \cdot 2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 18 & 8 \\ 11 & 5 \end{pmatrix}.$$

This is a matrix of dimension $3 \times 2$.

If the matrices $\mathcal{A}$ and $\mathcal{B}$ may be multiplied in any order, and if they are not square ones, then the product $\mathcal{A} \mathcal{B}$ and $\mathcal{B} \mathcal{A}$ will be square matrices of different dimensions.

**Exercise.** Find the products $\mathcal{A} \mathcal{B}$ and $\mathcal{B} \mathcal{A}$ of the following matrices:

$$\mathcal{A} = \begin{pmatrix} 1 & -2 & 0 \\ 3 & -1 & 5 \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} 1 & 7 \\ 3 & 4 \\ -1 & 0 \end{pmatrix}.$$
Solution:
Let us first compute $AB$:

$$AB = \begin{pmatrix} 1 - 6 + 0 & 7 - 8 + 0 \\ 3 - 3 - 5 & 21 - 4 + 0 \end{pmatrix} = \begin{pmatrix} -5 & -1 \\ -5 & 17 \end{pmatrix}. $$

If we now multiply the same matrices in reverse order then we get the matrix of dimension $3 \times 3$:

$$BA = \begin{pmatrix} 1 + 21 & -2 - 7 & 0 + 35 \\ 3 + 12 & -6 - 4 & 0 + 20 \\ -1 + 0 & 2 + 0 & 0 + 0 \end{pmatrix} = \begin{pmatrix} 22 & -9 & 35 \\ 15 & -10 & 20 \\ -1 & 2 & 0 \end{pmatrix}.  $$

Square matrices of the same dimension may be always multiplied in any order. In general, even for square matrices of the same dimension $AB \neq BA$. If for some matrices we have the equality $AB = BA$ then they are called \textit{commutative}.

Exercise. Check whether the following matrices are commutative

$$A = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}. $$

Solution:
We multiply the matrices $A$ and $B$ first in one then in another order and compare the results.

$$A \cdot B = \begin{pmatrix} 1 \cdot 3 + 2 \cdot 1 & 1 \cdot 2 + 2 \cdot 1 \\ 1.3 + (-1) \cdot 1 & 1.2 + (-1) \cdot 1 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 2 & 1 \end{pmatrix},  $$

$$B \cdot A = \begin{pmatrix} 3 \cdot 1 + 2 \cdot 1 & 3 \cdot 2 + 2 \cdot 1 \\ 1 \cdot 1 + 1 \cdot 1 & 1 \cdot 2 + 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 2 & 1 \end{pmatrix}. $$

Thus, the equality $AB = BA$ holds, therefore, these matrices are commutative.

1.2. Determinants

Every square matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} $$

is called a \textit{matrix of order $n$}. The \textit{determinant} of this matrix is denoted by $\det(A)$ and is defined as

$$\det(A) = \sum_{j=1}^{n} a_{ij} \text{sgn}(i,j) \prod_{k \neq j} a_{ik}, $$

where \text{sgn}(i,j) is the sign of the permutation $(i,j)$.

The determinant of a $1 \times 1$ matrix $A = (a_{11})$ is defined as $\det(A) = a_{11}$. For a $2 \times 2$ matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, the determinant is $\det(A) = a_{11}a_{22} - a_{12}a_{21}$. For a $3 \times 3$ matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, the determinant is $\det(A) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$
has the determinant, which is a number, computed in a certain way. There
are the following notations for the determinant:

\[ \Delta = \det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}. \]

The determinant of a matrix of dimension \(2 \times 2\)

\[ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \]

is computed by the formula given below:

\[ \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}. \]

**Exercise.** Compute the determinant of the matrix

\[ A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}. \]

**Solution:**

\[ \det A = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -1 - 6 = -7. \]

**Remark.** The determinant of a matrix of dimension \(2 \times 2\) is called
a second order determinant.

The determinant of a matrix of dimension \(3 \times 3\)

\[ A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \]

is the following number

\[ \det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} - 
- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{21}a_{12}a_{33}. \]

There are two simple ways to compute a third order determinant.
Adding rows or columns (fig. 1). To the given determinant we add bellow two upper rows or write on the right two first columns. We take with the sign $\leftarrow+\rightarrow$ the product of elements of the main diagonal, meaning that, which goes from the upper left corner to the lower right one, and also the products of elements, standing on lines, which are parallel to it. The products of the elements of the secondary diagonal and also of those, being on the lines, which are parallel to it, are taken with the sign $\leftarrow-\rightarrow$. The sum of these six products gives the determinant.

![Adding rows or columns](image)

The rule of triangles (fig. 2). The products of the elements of the main diagonal and those being in the corners of the triangles, shown in fig. 2, a, are taken with the sign $\leftarrow+\rightarrow$. The products of the elements of the secondary diagonal and those being in the corners of the triangles, shown in fig 2, b are taken with the sign $\leftarrow-\rightarrow$. The determinant equals to the sum of these products.

![The rule of triangles](image)

Exercise. Compute the determinants of the matrices

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 2 & 1 \\ 4 & 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 4 & -1 \\ 7 & 3 & 2 \\ 3 & 1 & -1 \end{pmatrix}.$$
\[
\begin{vmatrix}
1 & 3 & 5 \\
0 & 2 & 1 \\
4 & 1 & 2
\end{vmatrix} = 4 + 12 + 0 - 40 - 1 - 0 = -25;
\]
\[
\begin{vmatrix}
2 & 4 & -1 \\
7 & 3 & 2 \\
3 & 1 & -1
\end{vmatrix} = -12 + 24 - 7 + 9 - 4 + 56 = 66.
\]

The determinants of the higher orders are defined by induction. To do it we need first the following definitions.

**Definition.** The minor determinant of a determinant

\[
\begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
\]

for the element \(a_{ij}\) is denoted by \(M_{ij}\) and is determinant of \((n - 1)\)-th order, obtained from \(\det A\) by striking out the \(i\)-th row and the \(j\)-th column.

**Example.** Let \(A\) be a matrix of dimension \(4 \times 4\)

\[
A = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{pmatrix}.
\]

Then, for instance,

\[
M_{12} = \begin{vmatrix}
a_{21} & a_{23} & a_{24} \\
a_{31} & a_{33} & a_{34} \\
a_{41} & a_{43} & a_{44}
\end{vmatrix}
\]

and

\[
M_{32} = \begin{vmatrix}
a_{11} & a_{13} & a_{14} \\
a_{21} & a_{23} & a_{24} \\
a_{41} & a_{43} & a_{44}
\end{vmatrix}.
\]

**Definition.** The algebraic complement \(A_{ij}\) of the element \(a_{ij}\) of a matrix \(A\) is the correspondent minor determinant \(M_{ij}\) multiplied by \((-1)^{i+j}\).

Now assume that we know how to compute all the determinants of \((n - 1)\) order for some \(n\). For example, we have formulas for computation
of the second and third order determinant \((n = 3, 4)\). Then the \(n\)-th order determinant of a matrix \(A\) is equal to the following expression:

\[
det A = \sum_{j=1}^{n} a_{1j} A_{1j}.
\]

Let us check that the above formula holds for third order determinants. Let \(A\) be a matrix

\[
A = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}.
\]

According to the formula:

\[
det A = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} =
\]

\[
= a_{11} \cdot (-1)^{1+1} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12} \cdot (-1)^{1+2} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} +
\]

\[
+ a_{13} \cdot (-1)^{1+3} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) -
\]

\[
- a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) =
\]

\[
= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} -
\]

\[
- a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31},
\]

which is correct.

Remark. It is not hard to verify that \(\det A\) equals also to \(\sum_{j=1}^{n} a_{ij} A_{ij}\), \(i\) is a fixed natural number less or equal to \(n\).

### 1.3. Some properties of the determinants

All the properties that are discussed below hold for arbitrary determinants. To simplify the exposition, we prove them only for the third order determinants. For the second order determinants they are obvious.

Let us first explain what is the transposition of a matrix.

**Definition.** Let \(A\) be a matrix of dimension \(m \times n\). Then the transposed matrix is denoted by \(A^T\), is of dimension \(n \times m\), and we have the relation \(a_{ij}^T = a_{ji}\) for any \(i = 1, \ldots, n\) and \(j = 1, \ldots, m\). In other words, we swap, in cases of square matrices, each row with the column, having the same number.
**Property 1.** The transposition does not alter the determinant: \( \det \mathbf{A}^T = \det \mathbf{A} \), assuming that \( \mathbf{A} \) is a square matrix.

*Proof.* We simply compute both determinants:

\[
\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} - \\
- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{21}a_{12}a_{33},
\]

\[
\det \mathbf{A}^T = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{12}a_{23}a_{31} - \\
- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{21}a_{12}a_{33}.
\]

The expressions coincide, \( \det \mathbf{A} = \det \mathbf{A}^T \), which concludes the proof.

**Property 2.** If we swap any two rows (or columns) the the determinant changes the sign (i.e. is multiplied by \((-1)\)). For instance,

\[
\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.
\]

(Here we swap first and second rows).

One proves this property by direct computations. Another way is to use the expression for a determinant in terms of its minor determinants.

**Property 3.** If all the elements of some row (or columns) of a determinant are zeros then it equals to zero. For example,

\[
\begin{vmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} = 0.
\]

**Property 4.** If the elements of some row (column) of a determinant are proportional to the corresponding elements of another row (column) then the determinant is zero.
Proof.

\[ \Delta = \begin{vmatrix} a_{11} & \lambda a_{11} & a_{13} \\ a_{21} & \lambda a_{21} & a_{23} \\ a_{31} & \lambda a_{31} & a_{33} \end{vmatrix} = a_{11}\lambda a_{21}a_{33} + \lambda a_{11}a_{23}a_{31} + a_{21}\lambda a_{31}a_{13} - \\
- a_{13}\lambda a_{21}a_{31} - a_{11}a_{23}\lambda a_{31} - a_{21}\lambda a_{11}a_{33} = 0. \]

Property 5. Let us assume that each element of some row (column) is represented as a sum of two summands. Then the determinant can be also represented as a sum of two other ones. Only those elements of these two determinants differ from the elements of the original one that are in the row (column) under consideration. Such elements of the first determinant are equal to the corresponding first summands of the original one, those of the second one equal to the corresponding second summands

\[ \begin{vmatrix} a_{11} & b_{12} + c_{12} & a_{13} \\ a_{21} & b_{22} + c_{22} & a_{23} \\ a_{31} & b_{32} + c_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & b_{12} & a_{13} \\ a_{21} & b_{22} & a_{23} \\ a_{31} & b_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & c_{12} & a_{13} \\ a_{21} & c_{22} & a_{23} \\ a_{31} & c_{32} & a_{33} \end{vmatrix}. \]

Property 6. Common multiplier of the elements of some row (column) can be taken ≪ out of the determinant ≫:

\[ \begin{vmatrix} a_{11} & \lambda a_{11} & a_{13} \\ a_{21} & \lambda a_{21} & a_{23} \\ a_{31} & \lambda a_{31} & a_{33} \end{vmatrix} = \lambda \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \]

Property 7. A determinant does not change if to the elements of some row (column) one adds the corresponding elements of some other row (column), multiplied by an arbitrary common coefficient:

\[ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} + \lambda a_{11} & a_{13} \\ a_{21} & a_{22} + \lambda a_{21} & a_{23} \\ a_{31} & a_{32} + \lambda a_{31} & a_{33} \end{vmatrix}. \]

Proof. Using the property 5, we can represent the determinant as a sum of two ones. The second summand is zero according to the property 4:

\[ \begin{vmatrix} a_{11} & a_{12} + \lambda a_{11} & a_{13} \\ a_{21} & a_{22} + \lambda a_{21} & a_{23} \\ a_{31} & a_{32} + \lambda a_{31} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & \lambda a_{11} & a_{13} \\ a_{21} & \lambda a_{21} & a_{23} \\ a_{31} & \lambda a_{31} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \]
Property 8. The sum of the elements of some row (column) of a
determinant, multiplied by the algebraic complement of the corresponding
elements of another row (column), is zero. For example,

\[ a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0, \]

etc.

Exercise. Using the above properties, compute the following
determinants.

1. \[ \begin{vmatrix} 2 & 3 & 1 \\ -2 & 2 & 4 \\ 1 & 2 & -1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 3 & 4 \\ 0 & 5 & 5 \\ 0 & 5 & 5 \end{vmatrix} \]

Solution:
1. We first add the second row to the first one, then take 2 out of the
second row and, after that, add the third one to it:

\[ \begin{vmatrix} 2 & 3 & 1 \\ -2 & 2 & 4 \\ 1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 5 & 5 \\ 0 & 5 & 5 \\ 0 & 5 & 5 \end{vmatrix} = \ldots \]

Now we compute the determinant, using minor determinants of the first
column:

\[ \ldots = 2 \left( 0 \cdot \begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 5 & 5 \\ 2 & -1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 5 & 5 \\ 3 & 1 \end{vmatrix} \right) = 2 \cdot 1 \cdot (5 - 15) = -20. \]

2. We begin computation with the help of the property 5:

\[ \begin{vmatrix} 1 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & -2 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 + 1 & 4 \\ -1 & -2 + 0 & 3 \end{vmatrix} = \]
\[ = \begin{vmatrix} 1 & 2 & 4 \\ 1 & 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ -1 & 0 \end{vmatrix} = 0 + \begin{vmatrix} 1 & 0 & 3 \end{vmatrix}. \]

The first summand being equal zero due to the property 4. Next, we
decompose the last determinant, using the second column:

\[ \begin{vmatrix} 1 & 1 & 4 \\ 1 & 0 & 3 \\ -1 & 0 & 5 \end{vmatrix} = 1 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 1 & 3 \\ -1 & 5 \end{vmatrix} = -(5 + 3) = -8. \]
1.4. Systems of linear equations. Cramer’s rule

Let us consider the system of three linear equations

\[
\begin{align*}
 a_{11}x + a_{12}y + a_{13}z &= b_1, \\
 a_{21}x + a_{22}y + a_{23}z &= b_2, \\
 a_{31}x + a_{32}y + a_{33}z &= b_3.
\end{align*}
\]

(1)

Definition. A triple \((x_0, y_0, z_0)\) is called a solution of the system (1) if each equation holds, with the variables being substituted by the corresponding values \((x \text{ by } x_0, \ldots)\).

Depending on the coefficients of a system, it can have one solution, infinitely many solutions or no solutions at all.

Theorem 1 (Cramer). Let us consider the determinant of a system (1)

\[\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.\]

If it is not equal to zero then the system has the unique solution. Moreover, there are the following formulas:

\[x_0 = \frac{\Delta x}{\Delta}, \quad y_0 = \frac{\Delta y}{\Delta}, \quad z_0 = \frac{\Delta z}{\Delta},\]

where

\[\Delta x = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \quad \Delta y = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \quad \Delta z = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}.\]

Exercise. Solve the system, using Cramer’s formulas:

\[
\begin{align*}
 2x - y + z &= 1, \\
 x + 3y + 2z &= -11, \\
 -x + 5y - z &= -8.
\end{align*}
\]

Solution:

The determinant of system is

\[\Delta = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 3 & 2 \\ -1 & 5 & -1 \end{vmatrix}.\]
To compute this determinant, we add the second column to the first one, then we add the second column, multiplied by 2, to the first one, then we add the second column to the third one (see property 7)

\[
\Delta = \begin{vmatrix} 0 & -1 & 0 \\ 7 & 3 & 5 \\ 9 & 5 & 4 \end{vmatrix} = -1 \cdot (-1)^{1+2} \begin{vmatrix} 7 & 5 \\ 9 & 4 \end{vmatrix} = 1 \cdot (28 - 45) = -17.
\]

We decomposed the determinant by means of minor determinants of the first row. The determinant is not equal zero. Therefore the system has the unique solution which can be found by using the Cramer’s formulas.

We need to compute the auxiliary determinants \( \Delta x, \Delta y, \Delta z \).

\[
\Delta x = \begin{vmatrix} 1 & -1 & 1 \\ -11 & 3 & 2 \\ -8 & 5 & -1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 \\ -8 & 3 & 5 \\ -3 & 5 & 4 \end{vmatrix} = -1 \cdot (28 - 45) = -17.
\]

Here we added the second column to the first one and to the second one, then we decomposed the determinant using the first row.

\[
\Delta y = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -11 & 2 \\ -1 & -8 & -1 \end{vmatrix} = \begin{vmatrix} 0 & -15 & -1 \\ 0 & -19 & 1 \\ -1 & -8 & -1 \end{vmatrix} =
\]

\[-\begin{vmatrix} -15 & -1 \\ -19 & -1 \end{vmatrix} = -(-15 - 19) = 34.
\]

We added the third row, multiplied by 2, to the first one, then we added the third row to the second one after that we decomposed the determinant using the first column.

\[
\Delta z = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 3 & -11 \\ -1 & 5 & -8 \end{vmatrix} = \begin{vmatrix} 0 & 9 & -15 \\ 0 & 8 & -19 \\ -1 & 5 & -8 \end{vmatrix} =
\]

\[-\begin{vmatrix} 9 & -15 \\ 8 & -19 \end{vmatrix} = -(9 - 145) = 51.
\]

Here we used the same operations as in the computation of \( \Delta y \) plus, to simplify the calculation of the second order determinant, we subtracted the second row from the first one.

Now we can use the Cramer’s formulas:

\[
x_0 = \frac{\Delta x}{\Delta} = \frac{-17}{-17} = 1, \quad y_0 = \frac{\Delta y}{\Delta} = \frac{34}{-17} = -2, \quad z_0 = \frac{\Delta z}{\Delta} = \frac{51}{-17} = -3.
\]
Thus the triple \((1, -2, -3)\) is the unique solution of the system under consideration.

### 1.5. Inverse matrix and its application to the systems of linear equations

Another method to solve a system of \(n\) linear equations with \(n\) variables is given by the use of inverse matrix.

**Definition.** For a square matrix \(A\)

\[
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\]

a matrix \(B\) is called inverse, if the following relation hold: \(AB = BA = I\) where \(I\) is \(n\)-th order identity matrix. It is denoted by \(A^{-1}\).

**Remark.** It is easy to see that if the inverse matrix exists then it is unique.

**Theorem 2.** A square matrix \(A\) has the inverse matrix if and only if its determinant is not zero, \(\det A \neq 0\).

The inverse matrix is computed by the formula:

\[
A^{-1} = \frac{1}{\det A} \begin{pmatrix}
A_{11} & A_{21} & \cdots & A_{1n} \\
A_{12} & A_{22} & \cdots & A_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
A_{1n} & A_{2n} & \cdots & A_{nn}
\end{pmatrix},
\]

where \(A_{ij}\) is the algebraic complement of the element \(a_{ij}\) of the matrix \(A\).

**Exercise.** Check whether the matrix

\[
A = \begin{pmatrix}
3 & 5 & -2 \\
1 & -3 & 2 \\
6 & 7 & -3
\end{pmatrix}
\]

has the inverse. If so then compute it.
Solution:
We compute first the determinant of the matrix $A$

$$\det A = \begin{vmatrix} 3 & 5 & -2 \\ 1 & -3 & 2 \\ 6 & 7 & -3 \end{vmatrix} = 27 + 60 - 14 - 36 - 42 + 15 = 10,$$

using the rule of triangles.

Since the determinant of matrix $A$ is not zero ($\det A \neq 0$), there exists the inverse matrix $A^{-1}$. To find the inverse matrix $A^{-1}$ we compute the algebraic complements $A_{ij}$:

$$A_{11} = (-1)^2 \begin{vmatrix} -3 & 2 \\ 7 & -3 \end{vmatrix} = 9 - 14 = -5,$$
$$A_{21} = (-1)^3 \begin{vmatrix} 5 & -2 \\ 7 & -3 \end{vmatrix} = -(-15 + 14) = 1,$$
$$A_{31} = (-1)^4 \begin{vmatrix} 5 & -2 \\ -3 & 2 \end{vmatrix} = 10 - 6 = 4,$$
$$A_{12} = (-1)^3 \begin{vmatrix} 1 & 2 \\ 6 & -3 \end{vmatrix} = -(-3 - 12) = 15,$$
$$A_{22} = (-1)^4 \begin{vmatrix} 3 & -2 \\ 6 & -3 \end{vmatrix} = -9 + 12 = 3,$$
$$A_{32} = (-1)^5 \begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix} = -(6 + 2) = -8,$$
$$A_{13} = (-1)^4 \begin{vmatrix} 1 & -3 \\ 6 & 7 \end{vmatrix} = 7 + 18 = 25,$$
$$A_{23} = (-1)^5 \begin{vmatrix} 3 & 5 \\ 6 & 7 \end{vmatrix} = -(21 - 30) = 9,$$
$$A_{33} = (-1)^6 \begin{vmatrix} 3 & 5 \\ 1 & -3 \end{vmatrix} = -9 - 5 = -14.$$

Thus the inverse matrix $A^{-1}$ equals to

$$A^{-1} = \frac{1}{10} \begin{pmatrix} -5 & 1 & 4 \\ 15 & 3 & -8 \\ 25 & 9 & -14 \end{pmatrix} = \begin{pmatrix} -0.5 & 0.1 & 0.4 \\ 1.5 & 0.3 & -0.8 \\ 2.5 & 0.9 & -1.4 \end{pmatrix}.$$ 

Let us consider now a system of three linear equations with three
variables:
\[
\begin{align*}
    a_{11}x + a_{12}y + a_{13}z &= b_1, \\
    a_{21}x + a_{22}y + a_{23}z &= b_2, \\
    a_{31}x + a_{32}y + a_{33}z &= b_3.
\end{align*}
\]

We introduce the following notations:
\[
\begin{align*}
    \mathbf{A} &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, & \mathbf{X} &= \begin{pmatrix} x \\ y \\ z \end{pmatrix}, & \mathbf{B} &= \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.
\end{align*}
\]

The system then can be written in the form of one matrix equation
\[
\mathbf{AX} = \mathbf{B},
\]
\(\mathbf{A}\) is called the of the coefficients of the system, \(\mathbf{X}\) vector-column of variables, \(\mathbf{B}\) is the vector-column of constant terms.

let us multiply both parts of this equation by the matrix \(\mathbf{A}^{-1}\) on the left (we assume that \(\det \mathbf{A} \neq 0\)):
\[
\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B}.
\]
Since \(\mathbf{A}^{-1}\mathbf{AX} = \mathbf{EX} = \mathbf{X}\), we get the solution of the system
\[
\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}.
\]

This method of finding the solution of a system of linear equations is called the matrix method.

**Exercise.** Solve the following system of linear equations by means of the matrix method
\[
\begin{align*}
    x + 2y + z &= 4, \\
    3x - 5y + 3z &= 1, \\
    2x + 7y - z &= 8.
\end{align*}
\]

Solution:
Consider the matrix of the coefficients of our system:
\[
\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -5 & 3 \\ 2 & 7 & -1 \end{pmatrix}.
\]
Let us compute its determinant:

$$\det A = \begin{vmatrix} 1 & 2 & 1 \\ 3 & -5 & 3 \\ 2 & 7 & -1 \end{vmatrix} = 33 \neq 0.$$ 

Since the determinant is not zero there exists the inverse matrix that in this case equals to

$$A^{-1} = \frac{1}{33} \begin{pmatrix} -16 & 9 & 11 \\ 9 & -3 & 0 \\ 31 & -3 & -11 \end{pmatrix}.$$ 

The vector-column of constant terms is equal to

$$B = \begin{pmatrix} 4 \\ 1 \\ 8 \end{pmatrix}.$$ 

Thus

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = X = A^{-1}B = \frac{1}{33} \begin{pmatrix} -16 & 9 & 11 \\ 9 & -3 & 0 \\ 31 & -3 & -11 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ 8 \end{pmatrix} = \frac{1}{33} \begin{pmatrix} -64 + 9 + 88 \\ 36 - 3 + 0 \\ 124 - 3 - 88 \end{pmatrix} = \frac{1}{33} \begin{pmatrix} 33 \\ 33 \\ 33 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$ 

In other words, the solution is the triple \((1, 1, 1)\)

### 1.6. Gauss method

The above methods of solving the systems of linear equations (Cramer’s rule and the matrix method) can be used only if the number of equations equals to the number of variables, and the determinant of a system is not zero. Gauss method is a more general tool to solve the systems of linear equations, because it can be applied to any systems of \(m\) linear equations with \(n\) variables.

To simplify the exposition we describe this method only for a system of three linear equations with three variables:

$$\begin{align*}
 a_{11}x + a_{12}y + a_{13}z &= b_1, \\
 a_{21}x + a_{22}y + a_{23}z &= b_2, \\
 a_{31}x + a_{32}y + a_{33}z &= b_3.
\end{align*}$$
If \( a_{11} = 0 \) then we swap the equations so that the coefficient by the variable \( x \) in the first equation becomes nonzero. Therefore we may assume that \( a_{11} \) is not zero. We multiply the first equation by \( \frac{a_{21}}{a_{11}} \) and subtract it from the second one. After that we multiply the first equation by \( \frac{a_{31}}{a_{11}} \) and it from the third one. We get the following system which is equivalent to the original one, meaning that its solution coincide with those of the original system.

\[
\begin{align*}
& a_{11}x + a_{12}y + a_{13}z = b_1, \\
& \left( a_{22} - \frac{a_{21}}{a_{11}}a_{12} \right) y + \left( a_{23} - \frac{a_{21}}{a_{11}}a_{13} \right) z = b_2 - \frac{a_{21}}{a_{11}}b_1, \\
& \left( a_{32} - \frac{a_{31}}{a_{11}}a_{12} \right) y + \left( a_{33} - \frac{a_{31}}{a_{11}}a_{13} \right) z = b_3 - \frac{a_{31}}{a_{11}}b_1.
\end{align*}
\]

We point out that the second and third equations do not depend on the variable \( x \). We assume now that \( \left( a_{22} - \frac{a_{21}}{a_{11}}a_{12} \right) \neq 0 \) (if it is not so, we simply swap the second and third equations; if the third equation is also independent of \( y \) then the next step in Gauss method is void). We multiply the second equation by \( \frac{a_{11}\tilde{a}_{32} - a_{31}a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \) and subtract it from the third one. Now the third equation is already independent of the variables \( x \) and \( y \).

Thus the system looks as follows:

\[
\begin{align*}
& a_{11}x + a_{12}y + a_{13}z = b_1, \\
& \tilde{a}_{22}y + \tilde{a}_{23}z = \tilde{b}_2, \\
& \tilde{a}_{33}z = \tilde{b}_3,
\end{align*}
\]

where \( \tilde{a}_{22}, \tilde{a}_{23}, \tilde{a}_{33}, \tilde{b}_2, \tilde{b}_3 \) are the new coefficients and constant terms.

If the diagonal coefficients \( a_{11}, \tilde{a}_{22}, \tilde{a}_{33} \) are not zeroes then the system has the unique solution that can be found by solving the system from the bottom equation up.

If, while solving the system of equations, one of them becomes \( \ll 0 = 0 \gg \) then this equation can be skipped.

If one of the equations becomes \( \ll 0 = b_j \gg \) with \( b_j \) being different from zero then the system clearly has no solutions.

If in the modified system the number of variables is greater than the number of equations then some variables can be chosen as parameters so that the number of variables which are not parameters equals to the number of equations and the diagonal coefficients of the new system are different from zero. In this case the system has infinitely many solutions.
Exercise. Use Gauss method to solve the following system of equations:

\[
\begin{align*}
    x - y + 2z &= -3, \\
    2x + 3y + 3z &= 5, \\
    3x + 4y + z &= 10.
\end{align*}
\]

Solution:
We multiply the first equation by \(2\) and subtract it from the second one. Then we multiply the first equation by \(3\) and subtract it from the third one. We get the system

\[
\begin{align*}
    x - y + 2z &= -3, \\
    5y - z &= 11, \\
    7y - 5z &= 19.
\end{align*}
\]

Next we multiply the second equation by \(\frac{7}{5}\) and subtract it from third one. The modified system looks as follows:

\[
\begin{align*}
    x - y + 2z &= -3, \\
    5y - z &= 11, \\
    -\frac{18}{5}z &= \frac{18}{5}.
\end{align*}
\]

The number of variables equals to the number of equations, and the diagonal coefficients are different from zero. Hence this system has the unique solution which we find by solving the system from the bottom equation up. It follows from the bottom equation that \(z = -1\). In the first two equations we change the variable \(z\) by its value and simplify obtained system:

\[
\begin{align*}
    x - y &= -1, \\
    5y &= 10.
\end{align*}
\]

Again we solve first the bottom equation and find that \(y = 2\). Then in the first equation we change the variable \(y\) by its value and simplify it:

\[x = 1.\]

Thus the triple \((1, 2, -1)\) is the unique solution of the original system.

Exercise. Solve the system of equations

\[
\begin{align*}
    x - 2y + 3z - 4t &= 5, \\
    2x + 3y + z + t &= 4, \\
    -x + y - 2z - t &= 3.
\end{align*}
\]
Solution:
We use Gauss method. After the first step we get the following system:

\[
\begin{cases}
  x - 2y + 3z - 4t = 5, \\
  7y - 5z + 9t = -6, \\
  -y + z - 5t = 8.
\end{cases}
\]

It is convenient now to swap the second and third equations and proceed further, using the Gauss method. We obtain then the system:

\[
\begin{cases}
  x - 2y + 3z - 4t = 5, \\
  -y + z - 5t = 8, \\
  2z - 26t = 50.
\end{cases}
\]

In the modified system the number of variables is greater than the number of equations. Let us choose the variable \( t \) as a parameter: \( t = \lambda \). The diagonal coefficients of the system with parameter are not zeroes. Therefore we can solve it starting from the bottom equation

\[ z = 25 + 13\lambda. \]

In the first two equations we change the variable \( z \) by its value, which depends on the parameter, and simplify the obtained system:

\[
\begin{cases}
  x - 2y = -70 - 35\lambda, \\
  y = 17 + 8\lambda.
\end{cases}
\]

Finally we get

\[ x = -19\lambda - 39, \quad y = 8\lambda + 17, \quad z = 13\lambda + 25, \quad t = \lambda. \]

Since \( \lambda \) is an arbitrary number, the original system has infinitely many solution. Each 4-tuple \((-19\lambda_0 - 39, 8\lambda_0 + 17, 13\lambda_0 + 25, \lambda_0)\) is a solution where \( \lambda_0 \) being a certain number.

Exercise. Solve the system

\[
\begin{cases}
  2x - y + 2z = 1, \\
  3x + 4y + z = 2, \\
  7x + 2y + 5z = 3.
\end{cases}
\]
Solution:
We use again Gauss method. So, we multiply the first equation by $\ll 3/2 \gg$ and subtract it from the second one. After this we multiply the first equation by $\ll 7/2 \gg$ and subtract it from the third one. We get then following system:

$$
\begin{cases}
2x - y + 2z = 1, \\
\frac{11}{2} y - 2z = \frac{1}{2}, \\
\frac{11}{2} y - 2z = -\frac{1}{2}.
\end{cases}
$$

Let us subtract now the second equation from the third one:

$$
\begin{cases}
2x - y + 2z = 1, \\
\frac{11}{2} y - 2z = \frac{1}{2}, \\
0 = -1.
\end{cases}
$$

This system clearly has no solution. Therefore the original system has also no solutions.
2. Complex numbers

2.1. Main definitions

A complex number is a pair of real numbers: \( z = (a, b), \ a, b \in \mathbb{R} \). There is another way to represent a complex number:

\[ z = a + ib, \]

where the number \( i \) satisfy the relation \( i^2 = -1 \). At the moment, this may seem rather artificial but in the sequel it becomes quite meaningful.

The number \( a \) is called the real part of a complex number \( z \), \( b \) is called the imaginary part. The following notation are used:

\[ a = \text{Re} z, \quad b = \text{Im} z \quad \text{or} \quad a = \Re z, \quad b = \Im z. \]

The number \( i \) is called the imaginary unit, all the numbers of the form \( ib \) are called pure imaginary numbers.

Complex numbers can be viewed as points of the real plane. Each number \( z = a + ib \) corresponds to the unique point of the real plane (fig. 3) with coordinates \( (a, b) \), and v.v., every point of the real plane with coordinates \( (a, b) \) corresponds to the unique complex number \( z = a + ib \).

![Fig 3](image)

If we consider the points of the real plane \( \mathbb{R}^2 \) as complex numbers then this plane is called complex and is denoted by \( \mathbb{C} \). The horizontal axis (axis OX) is called then the real axis. It consists of the points of the form \( z = a + i0 = a \). In other words, a point of the real axis is a real number. The vertical axis (axis OY) consists of the points of the form \( z = 0 + ib = ib \), these are pure imaginary numbers. That is why this axis is called imaginary.
The module of a complex number \( z \) is the distance from the origin of coordinates to the point \( z \). The module of a number \( z = a + ib \) is denoted by \( |z| \) and is equal to

\[
|z| = \sqrt{a^2 + b^2}.
\]

It is clear that \( |z| \) is a real non-negative number, the equality \( |z| = 0 \) holds if and only if \( z = 0 + i \cdot 0 = 0 \).

The angle between the position vector of a point \( z \) and the real axis is called the argument of the complex number and is denoted by \( \text{Arg} \ z \).

Two complex numbers \( z_1 = a_1 + ib_1 \) and \( z_2 = a_2 + ib_2 \) are equal if and only if their real and imaginary parts coincide \( a_1 = a_2, b_1 = b_2 \). In this case \( |z_1| = |z_2| \), and \( \text{Arg}z_1 = \text{Arg}z_2 + 2\pi k, \) \( k \in \mathbb{Z} \) (note that the argument of a complex number is defined only up to a summand of the form \( 2\pi n, \) \( n \in \mathbb{Z} \)). It is convenient to denote by \( \text{arg} \ z \) the angle that belongs to the interval \([0; 2\pi)\). Then \( \text{Arg}z = \text{arg} \ z + 2\pi m, \) \( m \in \mathbb{Z} \).

Let us denote now the module of a complex number \( z = a + ib \) by \( r \): \( r = |z| \), and the argument by \( \varphi \): \( \varphi = \text{arg} \ z \). Then it is easy to see that \( a = r \cos \varphi, \) \( b = r \sin \varphi \). therefore

\[
z = r(\cos \varphi + i \sin \varphi).
\]

We say that the complex number \( z \) is written in trigonometrical form, whereas, when we use the notation \( z = a + ib \) then we say that \( z \) is represented in algebraic form. To pass from algebraic form to trigonometric one, we use the equalities:

\[
r = \sqrt{a^2 + b^2}, \quad \text{tg} \ \varphi = \tan \varphi = \frac{b}{a}
\]

We should also know in which quarter the complex number \( z \) lies.

Exercise. Represent on the complex plane the numbers

\[
 z_1 = 2i, \quad z_2 = 1 + i, \quad z_3 = i - \sqrt{3}, \quad z_4 = -3.
\]

Find the modules and the arguments of these numbers and write them in trigonometrical form.

Solution:

Let us represent on the complex plane the numbers \( z_1, z_2, z_3, z_4 \) as it is shown on fig. 4. The number \( z_1 = 0 + 2i \) lies on the imaginary axis. Its module equal \( r_1 = |z_1| = \sqrt{0^2 + 2^2} = 2 \), the angle between its position
vector and the axis OX is clearly $\pi/2$, $\varphi_1 = \arg z_1 = \pi/2$. Therefore trigonometrical form of $z_1$ looks as follows

$$z_1 = 2 \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right).$$

The number $z_2$ is in the first quarter because its real and imaginary parts are positive. The module and argument are equal

$$r_2 = |z_2| = \sqrt{1^2 + 1^2} = \sqrt{2},$$
$$\tan \varphi_2 = \frac{1}{1} = 1 \Rightarrow \varphi_2 = \arg z_2 = \frac{\pi}{4}.$$

Thus we can write now the number $z_2$ in trigonometrical form

$$z_2 = \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right).$$

The number $z_3$ is in the second quarter:

$$r_3 = |z_3| = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2,$$
$$\tan \varphi_3 = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3} \Rightarrow \varphi_3 = \arg z_3 = \frac{5}{6}\pi,$$

$$z_3 = 2 \left( \cos \frac{5}{6}\pi + i \sin \frac{5}{6}\pi \right).$$

Finally the number $z_4$ belong the real axis:

$$|z_4| = \sqrt{(-3)^2 + 1^2} = 3, \quad \varphi_4 = \arg z_4 = \pi,$$
$$z_4 = 3 (\cos \pi + i \sin \pi).$$
2.2. Arithmetic operations with complex numbers

The sum of two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ is the complex number

$$z = (a_1 + a_2) + i(b_1 + b_2).$$

Their difference is the complex number

$$w = (a_1 - a_2) + i(b_1 - b_2).$$

In geometrical representation adding and subtracting of complex numbers correspond to adding and subtracting of the position vectors of the given numbers.

The product of two numbers $z_1$ and $z_2$ is the complex number

$$z = (a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2).$$

This expression can be obtained if we formally multiply $(a_1 + ib_1)$ by $(a_2 + ib_2)$ and take into account that $i^2 = -1$.

To multiply complex numbers it is convenient to use their trigonometrical forms. Indeed,

$$z_1 \cdot z_2 = r_1(\cos \varphi_1 + i\sin \varphi_1) \cdot r_2(\cos \varphi_2 + i\sin \varphi_2) =$$

$$= r_1r_2(\cos \varphi_1 \cos \varphi_2 + i\sin \varphi_1 \cos \varphi_2 + i\sin \varphi_2 \cos \varphi_1 + i^2 \sin \varphi_1 \sin \varphi_2) =$$

$$= r_1r_2((\cos \varphi_1 \cos \varphi_2 - \sin \varphi_1 \sin \varphi_2) + i(\sin \varphi_1 \cos \varphi_2 + \cos \varphi_1 \sin \varphi_2)).$$

Recall that:

$$\cos(\varphi_1 + \varphi_2) = \cos \varphi_1 \cos \varphi_2 - \sin \varphi_1 \sin \varphi_2,$$

$$\sin(\varphi_1 + \varphi_2) = \sin \varphi_1 \cos \varphi_2 + \cos \varphi_1 \sin \varphi_2.$$

Thus we get the following formula:

$$z_1 \cdot z_2 = r_1r_2(\cos(\varphi_1 + \varphi_2) + \sin(\varphi_1 + \varphi_2)).$$

So, the module of the product of two complex numbers is the product of their modules, and its argument is the sum of their arguments.

Using this formula it is easy to raise a complex number $z = r(\cos \varphi + i\sin \varphi)$ to a positive integer power. For instance,

$$z^2 = z \cdot z = r \cdot r(\cos(\varphi + \varphi) + \sin(\varphi + \varphi)) = r^2(\cos 2\varphi + i \sin 2\varphi),$$

$$z^3 = z^2 \cdot z = r^2 \cdot r(\cos(2\varphi + \varphi) + \sin(2\varphi + \varphi)) = r^3(\cos 3\varphi + i \sin 3\varphi).$$
In general for any positive integer number $n$ we have

$$z^n = r^n (\cos n\varphi + i \sin n\varphi).$$

This equality is called *Moivre’s formula*.

The number $\overline{z} = a - ib$ is called *conjugate* to $z = a + ib$. It is easy to see that

$$z \cdot \overline{z} = a^2 + b^2 = |z|^2,$$

i. e. it is real non-negative number. We note also that $|\overline{z}| = |z|$, $\arg(\overline{z}) = 2\pi - \arg(z)$. In geometric sense the conjugate number is symmetric to the given complex number with respect to the real axis.

We can define now the *inverse* of a complex number $z$: $z^{-1} = \frac{\overline{z}}{|z|^2}$.

Indeed,

$$z \cdot z^{-1} = z \cdot \frac{\overline{z}}{|z|^2} = \frac{1}{|z|^2} \cdot z \cdot \overline{z} = 1.$$

Furthermore it is clear how to define the quotient of two complex numbers:

$$\frac{z_1}{z_2} = z_1 \cdot z_2^{-1} = \frac{z_1 \cdot \overline{z}_2}{|z_2|^2} = \frac{(a_1 + ib_1)(a_2 - ib_2)}{a_2^2 + b_2^2} = \frac{(a_1a_2 + b_1b_2) + i(a_2b_1 - a_1b_2)}{a_2^2 + b_2^2}.$$

Thus

$$\frac{z_1}{z_2} = \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + i\frac{a_2b_1 - a_1b_2}{a_2^2 + b_2^2}.$$

where $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$.

It is much easier to divide complex numbers if they are represented in trigonometrical forms:

$$z = \frac{z_1}{z_2} = \frac{r_1 (\cos \varphi_1 + i \sin \varphi_1)}{r_2 (\cos \varphi_2 + i \sin \varphi_2)} = \frac{r_1}{r_2} \left( \cos (\varphi_1 - \varphi_2) + i \sin (\varphi_1 - \varphi_2) \right).$$

**Exercise.** Perform operations:

a. $(1 + i)(i - \sqrt{3})$;

b. $\frac{i - \sqrt{3}}{1 + i}$;

c. $(1 + i)^7$. 


Solution.

a. We first compute the product in algebraic form:

\[(1 + i)(i - \sqrt{3}) = -\sqrt{3} - i\sqrt{3} + i + i^2 = -\sqrt{3} - i\sqrt{3} + i - 1 =
\]
\[= -(\sqrt{3} + 1) + i(1 - \sqrt{3}).\]

The same product can be also computed in trigonometrical form:

\[(1 + i)(i - \sqrt{3}) = \sqrt{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) \cdot 2 \left(\cos \frac{5}{6}\pi + i \sin \frac{5}{6}\pi\right) =
\]
\[= 2\sqrt{2}\left(\cos \left(\frac{\pi}{4} + \frac{5}{6}\pi\right) + i \sin \left(\frac{\pi}{4} + \frac{5}{6}\pi\right)\right) =
\]
\[= 2\sqrt{2}\left(\cos \frac{13}{12}\pi + i \sin \frac{13}{12}\pi\right).
\]

b. Let us calculate this fraction in algebraic form:

\[
\frac{i - \sqrt{3}}{1 + i} = \frac{(i - \sqrt{3})(1 - i)}{(1 + i)(1 - i)} = \frac{-\sqrt{3} + i + i\sqrt{3} - i^2}{1 + 1} =
\]
\[= \frac{(-\sqrt{3} + 1) + i(\sqrt{3} + 1)}{2} = \frac{-\sqrt{3} + 1}{2} + i\frac{\sqrt{3} + 1}{2}.
\]

Then we compute it in trigonometrical form

\[
\frac{i - \sqrt{3}}{1 + i} = \frac{2\left(\cos \frac{5}{6}\pi + i \sin \frac{5}{6}\pi\right)}{\sqrt{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)} =
\]
\[= \sqrt{2}\left(\cos \left(\frac{5}{6}\pi - \frac{\pi}{4}\right) + i \sin \left(\frac{5}{6}\pi - \frac{\pi}{4}\right)\right) =
\]
\[= \sqrt{2}\left(\cos \frac{7}{12}\pi + i \sin \frac{7}{12}\pi\right).
\]

c. To find the power of the complex number, we first represent it in trigonometrical form and then use Moivre’s formula:

\[(1 + i)^7 = \left(\sqrt{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)\right)^7 = \left(\sqrt{2}\right)^7 \left(\cos \frac{7}{4}\pi + i \sin \frac{7}{4}\pi\right) =
\]
\[= \left(\sqrt{2}\right)^7 \left(\cos \left(2\pi - \frac{\pi}{4}\right) + i \sin \left(2\pi - \frac{\pi}{4}\right)\right) =
\]
\[
= \left( \sqrt{2} \right)^7 \left( \cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right) = \left( \sqrt{2} \right)^7 \left( \frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right) = \left( \sqrt{2} \right)^7 \frac{1}{\sqrt{2}} (1 - i) = \left( \sqrt{2} \right)^6 (1 - i) = 2^3 (1 - i) = 8 (1 - i) = 8 - 8i.
\]

2.3. Extracting a root from a complex number

A complex number \( w \) is called a \( n \)-th root of a complex number \( z \), and is denoted by \( n\sqrt{z} \), if \( z = w^n \). It is convenient to find the roots of a complex number when it is represented in trigonometrical form. Let \( z = r(\cos \varphi + i \sin \varphi) \), and \( w = \rho(\cos \theta + i \sin \theta) \), where \( \rho \) and \( \theta \) are to be found. Moivre’s formula implies then that

\[
w^n = \rho^n (\cos n\theta + i \sin n\theta).
\]

Since the numbers \( z \) and \( w^n \) are equal we have the following relations:

\[
\rho^n = r, \quad n\theta_k = \varphi + 2\pi k,
\]

where \( k \) is an arbitrary integer number. It follows that

\[
\rho = \sqrt[n]{r}, \quad \theta_k = \frac{\varphi}{n} + \frac{2\pi k}{n},
\]

We see that the arguments depends on \( k \). However, since the functions cosine and sine are \( 2\pi \)-periodic, we get only \( n \) different roots, when, for example, \( k \) differs from 0 to \( n - 1 \).

Thus, to find \( n \) values of \( n \)-th root of a complex number, we have the following formulas:

\[
\sqrt[n]{z} = \rho(\cos \theta_k + i \sin \theta_k),
\]

\[
\rho = \sqrt[n]{r}, \quad \theta_k = \frac{\varphi}{n} + \frac{2\pi k}{n}, \quad k = 0, 1, 2, \ldots, n - 1,
\]

where \( z = r(\cos \varphi + i \sin \varphi) \).

It is clear now that the points of the complex plane, corresponding to different values of \( n \)-th root of a complex number, are the vertices of a regular polygon, inscribed in the circle with the radius \( \sqrt[n]{r} \) whose center is in the origin of coordinates.

Exercise. Find all the values \( \sqrt[3]{i} \).
Solution:
We first represent the number $i$ in trigonometrical form

\[ i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}. \]

We have therefore in the above notation $r = 1$ and $\varphi = \frac{\pi}{2}$. Now we use the obtained formula to write down all the values of the cube root:

\[ \rho = \sqrt[3]{1} = 1, \quad \theta_k = \frac{\pi/2}{3} + \frac{2\pi k}{3}, \quad k = 0, 1, 2. \]

Thus, we get the following three roots:

- for $k = 0$ \[ w_0 = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{i}{2}; \]
- for $k = 1$ \[ w_1 = \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{i}{2}; \]
- for $k = 2$ \[ w_2 = \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} = -i. \]

\[ 2.4. \text{ Quadratic equations} \]

We consider in this section monic quadratic equations with complex coefficients (the coefficient by $z^2$ is equal to 1):

\[ z^2 + a_1 z + a_2 = 0, \]

where $a_1, a_2 \in \mathbb{C}$, i.e. $a_1, a_2$ are complex numbers. The question is how to find the roots of such an equation. Using arithmetic operations with complex numbers one easily verifies that the usual formula works:

\[ z_1 = -\frac{a_1}{2} + \sqrt{\frac{a_1^2}{4} - a_2}, \quad z_2 = -\frac{a_1}{2} + \sqrt{\frac{a_1^2}{4} - a_2}, \quad (2) \]

where we denote by $(\sqrt{\_})_1$ and $(\sqrt{\_})_2$ the two values of a square root of a complex number.

We know already how to extract a square root from a complex number (see the section 2.3). However, sometimes it is more convenient to use another way. So, let $z$ be a complex number $z = a + ib$. Let its square
root be \( w = c + id \), where \( c \) and \( d \) are the real numbers to be found. From equation \( w^2 = z \) we have
\[
a + ib = (c + id)^2 = (c^2 - d^2) + i(2cd).
\]
Two complex numbers equal if and only if their real and imaginary parts coincide. Hence we get the system of equations
\[
\begin{aligned}
c^2 - d^2 &= a, \\
2cd &= b.
\end{aligned}
\]
We may assume that \( d \neq 0 \) because, otherwise, the number \( z \) is in fact real and this case being of no interest. Solving this system for \( d \neq 0 \) we have
\[
d_{1,2} = \pm \sqrt{-a + \sqrt{a^2 + b^2}} = c_j = \frac{b}{2d_j}, \quad j = 1, 2.
\]
Therefore \( w_1 = c_1 + id_1 \) and \( w_2 = c_2 + id_2 \) are the two values of a square root of the complex number \( s = a + ib \).

**Exercise.** Find the roots of the following quadratic equation:
\[
z^2 + (i - 5)z + 8 - i = 0.
\]

**Solution:**
By the formula (2)
\[
z_{1,2} = \frac{5 - i}{2} + \left( \sqrt{\frac{D}{4}} \right)_{1,2}
\]
where
\[
\frac{D}{4} = \frac{(5 - i)^2}{4} - 8 + i = \frac{25 - 10i + i^2 - 32 + 4i}{4} = \frac{8 - 6i}{4} = -2 - \frac{3}{2}i.
\]
We use just described method: \( w_j = \left( \sqrt{\frac{D}{4}} \right)_j = c_j + id_j, \quad (j = 1, 2) \),
where
\[
d_1 = \sqrt{\frac{2 + \sqrt{5}/2}{2}} = \sqrt{\frac{9}{4}} = \frac{3}{2}, \quad c_1 = \frac{-3/2}{2 \cdot 3/2} = -\frac{1}{2};
\]
\[
d_2 = -d_1 = -\frac{3}{2}, \quad c_2 = -c_1 = \frac{1}{2}.
\]
Hence
\[
z_1 = \frac{5 - i}{2} + \frac{-1 + 3i}{2} = 2 + i; \quad z_1 = \frac{5 - i}{2} + \frac{1 - 3i}{2} = 3 - 2i
\]
3. A sample of the test with solution

Variant 26
I. Find the inverse matrix for $A$.

$$A = \begin{pmatrix}
1 & 2 & 3 \\
-1 & 0 & 3 \\
-1 & -2 & 0
\end{pmatrix}. $$

Solution:
If $\det A \neq 0$ then the inverse matrix $A^{-1}$ exists. So we first compute the determinant

$$\det A = \Delta A = \begin{vmatrix}
1 & 2 & 3 \\
-1 & 0 & 3 \\
-1 & -2 & 0
\end{vmatrix} = 3 \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 6.$$

Then we calculate the algebraic complements of matrix $A$:

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 0 & 3 \\ -2 & 0 \end{vmatrix} = 6, \quad A_{12} = (-1)^{1+2} \begin{vmatrix} -1 & 3 \\ -1 & 0 \end{vmatrix} = -3,$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} -1 & 0 \\ -1 & -2 \end{vmatrix} = 2, \quad A_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = -6,$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} = 3, \quad A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ -1 & -2 \end{vmatrix} = 0,$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 3 \\ 0 & 3 \end{vmatrix} = 6, \quad A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ -1 & 3 \end{vmatrix} = -6,$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 2.$$

Using the formula for inverse matrix we get

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix}
A_{11} & A_{21} & A_{31} \\
A_{12} & A_{22} & A_{32} \\
A_{13} & A_{23} & A_{33}
\end{pmatrix} =$$

$$= \frac{1}{6} \begin{pmatrix}
6 & -6 & 6 \\
-3 & 3 & -6 \\
2 & 0 & 2
\end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\
-1/2 & 1/2 & -1 \\
1/3 & 0 & 1/3
\end{pmatrix}.$$

Answer: $A^{-1} = \begin{pmatrix}
1 & -1 & 1 \\
-1/2 & 1/2 & -1 \\
1/3 & 0 & 1/3
\end{pmatrix}.$
II. Solve the system of linear equations, using first Cramer’s formulas then the matrix method and then Gauss method.

\[
\begin{align*}
\begin{cases}
  x + y - z &= 1, \\
  8x + 3y - 6z &= 2, \\
  -4x - y + 3z &= -3.
\end{cases}
\end{align*}
\]

**Cramer’s formulas:** We compute the determinant of the system:

\[
\Delta = \begin{vmatrix}
  1 & 1 & -1 \\
  8 & 3 & -6 \\
  -4 & -1 & 3
\end{vmatrix} = \begin{vmatrix}
  1 & 0 & 0 \\
  8 & -5 & 2 \\
  -4 & 3 & -1
\end{vmatrix} = \begin{vmatrix}
  -5 & 2 \\
  3 & -1
\end{vmatrix} = 5 - 6 = -1.
\]

The determinant is not zero, therefore the system has the unique solution that can be found by means of Cramer’s formulas. Thus we need to compute the auxiliary determinants:

\[
\Delta_x = \begin{vmatrix}
  1 & 1 & -1 \\
  2 & 3 & -6 \\
  -3 & -1 & 3
\end{vmatrix} = \begin{vmatrix}
  1 & 0 & 0 \\
  2 & 1 & -4 \\
  -3 & 2 & 0
\end{vmatrix} = \begin{vmatrix}
  1 & -4 \\
  2 & 0
\end{vmatrix} = 0 + 8 = 8,
\]

\[
\Delta_y = \begin{vmatrix}
  1 & 1 & -1 \\
  8 & 2 & -6 \\
  -4 & -3 & 3
\end{vmatrix} = \begin{vmatrix}
  1 & 0 & 0 \\
  8 & -6 & 2 \\
  -4 & 1 & -1
\end{vmatrix} = \begin{vmatrix}
  -6 & 2 \\
  1 & -1
\end{vmatrix} = 6 - 2 = 4,
\]

\[
\Delta_z = \begin{vmatrix}
  1 & 1 & 1 \\
  8 & 3 & 2 \\
  -4 & -1 & -3
\end{vmatrix} = \begin{vmatrix}
  1 & 0 & 0 \\
  8 & -5 & -6 \\
  -4 & 3 & 1
\end{vmatrix} = \begin{vmatrix}
  -5 & -6 \\
  3 & 1
\end{vmatrix} = -5 + 18 = 13.
\]

Now we apply Cramer’s formulas and find the solution of the system:

\[
x = \frac{\Delta_x}{\Delta} = \frac{8}{-1} = -8,
\]

\[
y = \frac{\Delta_y}{\Delta} = \frac{4}{-1} = -4,
\]

\[
z = \frac{\Delta_z}{\Delta} = \frac{13}{-1} = -13.
\]

Thus the triple \((-8, -4, -13)\) is the unique solution of the system.

**Matrix method:** Any system of linear equations can be written in matrix form: \(AX = B\). In our case this equation looks as follows:

\[
\begin{pmatrix}
  1 & 1 & -1 \\
  8 & 3 & -6 \\
  -4 & -1 & 3
\end{pmatrix}
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} =
\begin{pmatrix}
  1 \\
  2 \\
  -3
\end{pmatrix}.
\]
The determinant of the system is not zero then the inverse matrix exists, and therefore it follows that
\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 8 & 3 & -6 \\ -4 & -1 & 3 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}.
\]
The inverse matrix is equal
\[
\begin{pmatrix} 1 & 1 & -1 \\ 8 & 3 & -6 \\ -4 & -1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -3 & 2 & 3 \\ 0 & 1 & 2 \\ -4 & 3 & 5 \end{pmatrix}.
\]
So
\[
X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 & 2 & 3 \\ 0 & 1 & 2 \\ -4 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -8 \\ -4 \\ -13 \end{pmatrix}.
\]
Hence the triple \((-8, -4, -13)\) is the unique solution of the system.

**Gauss method:** We multiply the first equation of the system by \(8\) and subtract it from the second one. Then we multiply the same equation by \(4\) and add it to third one. We get the following system which is equivalent to the given one:
\[
\begin{cases}
x + y - z = 1, \\
-5y + 2z = -6, \\
3y - z = 1.
\end{cases}
\]
The next step in Gauss method is multiply the second equation by \(3/5\) and add it to last one. After that the system looks as follows:
\[
\begin{cases}
x + y - z = 1, \\
-5y + 2z = -6, \\
\frac{1}{5}z = -\frac{13}{5}.
\end{cases}
\]
The obtained system is easily solved from the bottom equation equation up.
\[
\begin{cases}
x + y - z = 1, \\
-5y + 2z = -6, \\ z = -13, \\
x + y = -12, \\ y = -4, \\ z = -13.
\end{cases} \Rightarrow \begin{cases}
x = -8, \\ y = -4, \\ z = -13.
\end{cases}
\]
Thus the triple \((-8, -4, -13)\) is the unique solution of the system.
III. Compute
\[
\frac{(-1 + i\sqrt{3})^{15}}{2^{10}}.
\]

Solution:
We represent the complex number \(-1 + i\sqrt{3}\) in trigonometrical form
\[
-1 + i\sqrt{3} = 2 \left( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)
\]
and use Moivre’s formula
\[
\frac{(-1 + i\sqrt{3})^{15}}{2^{10}} = \frac{\left( 2 \left( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) \right)^{15}}{2^{10}} = \frac{2^{15} (\cos 10\pi + i \sin 10\pi)}{2^{10}} = 2^5 = 32.
\]

Answer: 32.

IV. Find all the values \(\sqrt[3]{i}\).

Solution:
Let us represent the complex number \(i\) in trigonometrical form
\[
i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}.
\]
The module of this number is equal to \(r = 1\), the argument equals \(\frac{\pi}{2}\). Thus if \(\sqrt[3]{i} = \rho (\cos \theta + i \sin \theta)\) then
\[
\rho = \sqrt[3]{1} = 1,
\theta_k = \frac{\pi}{6} + \frac{2\pi k}{3}, \quad k = 0, 1, 2.
\]
So we get three values of the cube root of the number \(i\):

for \(k = 0\) \quad \(w_0 = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{i}{2}\);

for \(k = 1\) \quad \(w_1 = \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{i}{2}\);

for \(k = 2\) \quad \(w_2 = \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} = -i.\)
The test. Linear algebra and complex numbers

I. Find the inverse matrix for $A$.

1. \[
\begin{pmatrix}
0 & 3 & 4 \\
-2 & 0 & 3 \\
-2 & -3 & 0
\end{pmatrix};
\]
2. \[
\begin{pmatrix}
0 & 3 & 4 \\
-3 & 0 & 4 \\
-3 & -4 & 0
\end{pmatrix};
\]
3. \[
\begin{pmatrix}
-1 & -2 & 0 \\
-1 & -2 & -3 \\
-1 & 0 & 3
\end{pmatrix};
\]

4. \[
\begin{pmatrix}
1 & 2 & 2 \\
2 & 2 & 2 \\
2 & 2 & 3
\end{pmatrix};
\]
5. \[
\begin{pmatrix}
1 & -2 & -2 \\
1 & 2 & 2 \\
2 & -2 & -3
\end{pmatrix};
\]
6. \[
\begin{pmatrix}
1 & 3 & 3 \\
3 & 3 & 3 \\
3 & 3 & 4
\end{pmatrix};
\]

7. \[
\begin{pmatrix}
2 & 1 & 2 \\
2 & 1 & -1 \\
7 & 2 & -3
\end{pmatrix};
\]
8. \[
\begin{pmatrix}
2 & 1 & -1 \\
7 & 2 & -3 \\
7 & 6 & -6
\end{pmatrix};
\]
9. \[
\begin{pmatrix}
1 & 1 & -3 \\
1 & -1 & -3 \\
1 & -1 & -6
\end{pmatrix};
\]

10. \[
\begin{pmatrix}
5 & 9 & 7 \\
5 & 3 & -2 \\
0 & 2 & -1
\end{pmatrix};
\]
11. \[
\begin{pmatrix}
4 & 0 & 5 \\
7 & -2 & 9 \\
3 & 0 & 6
\end{pmatrix};
\]
12. \[
\begin{pmatrix}
1 & -2 & 3 \\
1 & -1 & 2 \\
0 & 1 & 1
\end{pmatrix};
\]
13. \[
\begin{pmatrix}
2 & -1 & -1 \\
-1 & 2 & -3 \\
-1 & 1 & -2
\end{pmatrix};
\]
14. \[
\begin{pmatrix}
-2 & 3 & -4 \\
-1 & 2 & -3 \\
-1 & 1 & -2
\end{pmatrix};
\]
15. \[
\begin{pmatrix}
1 & -1 & 16 \\
0 & 1 & -1 \\
0 & 1 & 3
\end{pmatrix};
\]

16. \[
\begin{pmatrix}
3 & 0 & -1 \\
1 & 3 & -1 \\
1 & 0 & 2
\end{pmatrix};
\]
17. \[
\begin{pmatrix}
-1 & 1 & 1 \\
-1 & 1 & 2 \\
-1 & 1 & 2
\end{pmatrix};
\]
18. \[
\begin{pmatrix}
2 & 0 & 3 \\
-1 & 3 & -4 \\
1 & 2 & -1
\end{pmatrix};
\]

19. \[
\begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & -1 \\
-1 & 1 & 1
\end{pmatrix};
\]
20. \[
\begin{pmatrix}
-1 & 11 & 7 \\
0 & 5 & -4 \\
0 & 1 & 1
\end{pmatrix};
\]
21. \[
\begin{pmatrix}
-2 & 1 & -2 \\
-1 & 3 & 2 \\
-3 & 1 & 2
\end{pmatrix};
\]

22. \[
\begin{pmatrix}
1 & 3 & -1 \\
0 & 1 & -1 \\
-1 & 1 & 1
\end{pmatrix};
\]
23. \[
\begin{pmatrix}
1 & 7 & 9 \\
0 & 5 & -4 \\
-1 & 3 & 1
\end{pmatrix};
\]
24. \[
\begin{pmatrix}
0 & 1 & 2 \\
-5 & 2 & 4 \\
10 & 1 & -1
\end{pmatrix};
\]

25. \[
\begin{pmatrix}
5 & -2 & 0 \\
-3 & 1 & 2 \\
9 & 7 & -1
\end{pmatrix};
\]
26. \[
\begin{pmatrix}
1 & 2 & 3 \\
-1 & 0 & 3 \\
-1 & -2 & 0
\end{pmatrix}.
II. Solve the system of linear equations, using first Cramer’s formulas then the matrix method and then Gauss method.

\[
\begin{align*}
1. & \begin{cases}
5x + 8y - z = 7, \\
2x - 3y + 2z = 9, \\
x + 2y + 3z = 1;
\end{cases} & \quad 2. & \begin{cases}
x + y - z = 2, \\
4x - 3y + z = 1, \\
2x + y = 5;
\end{cases} \\
3. & \begin{cases}
2x - y + 5z = 4, \\
5x + 2y + 13z = -23, \\
3x - z = -5;
\end{cases} & \quad 4. & \begin{cases}
x - 4y - 2z = -3, \\
3x + y + z = 5, \\
-3x + 5y + 6z = 7;
\end{cases} \\
5. & \begin{cases}
7x - 5y = 31, \\
4x + 11z = -43, \\
x + y - z = 4, \\
x + y - z = 4; \\
2x + 3y + 4z = -20; \\
8x + 3y - 6z = 24, \\
-4x - y + 3z = -11; \\
4x - 3y + 2z = 9, \\
2x + 5y - 3z = 4, \\
5x + 6y - 2z = 18; \\
7x - 5y = 11, \\
4x + 11z = 23, \\
2x + 3y + 4z = 16; \\
2x - y + 5z = -8, \\
5x + 2y + 13z = 2, \\
3x - z = 4; \\
x + 2y + z = 5, \\
3x - 5y + 3z = 1, \\
2x + 7y - z = 14; \\
4x - 3y + 2z = 4, \\
2x + 5y - 3z = 19, \\
5x + 6y - 2z = 29; \\
5x + 8y - z = 7, \\
x + 2y + 3z = 1, \\
2x - 3y + 2z = 9; \\
\end{cases} & \quad 10. & \begin{cases}
2x + y - z = 1, \\
x + y - z = 1, \\
4x - 3y + z = 1; \\
2x + y - z = 1, \\
x + y - z = -2, \\
4x - 3y + z = 1; \\
5x + 8y - z = 28, \\
2x - 3y + 2z = -5, \\
x + 2y + 3z = 10; \\
x - 4y - 2z = -3, \\
3x + y + z = 3, \\
-3x + 5y + 6z = 34; \\
x - 2y + 3z = 6, \\
2x + 3y - 4z = 20, \\
3x - 2y - 5z = 6; \\
\end{cases} \\
12. & \begin{cases}
2x - 3y + 2z = -5, \\
x + 2y + 3z = 10; \\
\end{cases} & \quad 14. & \begin{cases}
3x + y + z = 3, \\
-3x + 5y + 6z = 34; \\
\end{cases} \\
16. & \begin{cases}
2x + 3y - 4z = 2, \\
3x - 2y - 5z = -8; \\
\end{cases} & \quad 18. & \begin{cases}
2x + y - z = -2, \\
x + y - z = -1, \\
4x - 3y + z = -8; \\
\end{cases} \\
19. & \begin{cases}
2x - y + 5z = 4, \\
2x + 3y + 13z = -23, \\
3x - z = -5; \\
\end{cases}
\end{align*}
\]
21. \(\begin{align*}
  x + y - z &= 1, \\
  8x + 3y - 6z &= 2, \\
  -4x - y + 3z &= -3; \\
  x + 2y + 3z &= 10, \\
  3x + 2y + z &= 4, \\
  y - z &= -1; \\
  3x + 2y + z &= 8, \\
  -x + 2y - z &= 0, \\
  x - y + z &= 1; \\
\end{align*}\)

22. \(\begin{align*}
  x + y - z &= -2, \\
  4x - 3y + z &= 1, \\
  2x + y - z &= 1; \\
  x - 2z &= -7, \\
  x + 2y + 3z &= 8, \\
  -x + 2y - z &= -2; \\
  x + y - z &= 1, \\
  8x + 3y - 6z &= 2, \\
  -4x - y + 3z &= -3. \\
\end{align*}\)

III. Compute.

1. \(\left(\frac{2 + i\sqrt{12}}{8}\right)^{24} \frac{16}{8};\)

2. \(\frac{(1 + i)^{18}}{64};\)

3. \(\frac{(\sqrt{3} - i)^{20}}{86};\)

4. \(\frac{(1 - i)^{20}}{512};\)

5. \(\frac{(i + i^2)^{12}}{16};\)

6. \(\frac{(i^3 - 1)^{10}}{4};\)

7. \(\frac{\left(\sqrt{3}i^2 + i\right)^{11}}{4^4};\)

8. \(\frac{(i^4 + i)^{22}}{16^2};\)

9. \(\frac{(\sqrt{3}i^4 + i^3)^{14}}{32^2};\)

10. \(\frac{(\sqrt{12} + 2i)^{15}}{8^{10}};\)

11. \(\frac{(i^2 + i)^{16}}{256};\)

12. \(\frac{(1 + \sqrt{3}i)^{27}}{8^8};\)

13. \(\frac{(i^2 - i^3)^{24}}{8^3};\)

14. \(\frac{(i^4 + i^5)^{18}}{16^2};\)

15. \(\frac{(i^4 - i)^{20}}{8^3};\)

16. \(\frac{(\sqrt{3}i^3 - 1)^{15}}{4^6};\)

17. \(\frac{(i^5 - 1)^{30}}{8^4};\)

18. \(\frac{(\sqrt{3}i^5 + i^4)^{30}}{16^4};\)

19. \(\frac{(2\sqrt{3}i - 2)^{12}}{4^{10}};\)

20. \(\frac{(3i - 3)^{22}}{18^{10}};\)

21. \(\frac{(i - \sqrt{3})^{36}}{16^{8}};\)

22. \(\frac{(i^2 + i)^{15}}{128\sqrt{2}};\)

23. \(\frac{(i^7 - i^8)^{16}}{8^3};\)

24. \(\frac{(i^5 + 1)^{22}}{4^5};\)

25. \(\frac{(3\sqrt{3}i - 3)^{18}}{36^9};\)

26. \(\frac{(-1 + i\sqrt{3})^{15}}{2^{10}}.\)

IV. Find all the values.

1. \(\frac{4}{\sqrt{1}};\)

2. \(\frac{4}{\sqrt{i}};\)

3. \(\frac{4}{\sqrt{-1}};\)

4. \(\frac{4}{\sqrt{1 - i}};\)

5. \(\frac{4}{\sqrt{-2 + 2i}};\)

6. \(\sqrt{1 - i};\)
7. $\sqrt[6]{-8}$; 8. $\sqrt[8]{1}$; 9. $\sqrt[9]{1+i}$;
10. $\sqrt[3]{-1}$; 11. $\sqrt[3]{1-i}$; 12. $\sqrt[3]{-2+2i}$;
13. $\sqrt[3]{1+i}$; 14. $\sqrt[3]{-i}$; 15. $\sqrt[3]{-1}$;
16. $\sqrt[5]{i}$; 17. $\sqrt[5]{1}$; 18. $\sqrt[5]{1+i}$;
19. $\sqrt[6]{1}$; 20. $\sqrt[3]{1-i\sqrt{3}}$; 21. $\sqrt[3]{\sqrt{3}+i}$;
22. $\sqrt[3]{-\sqrt{3}+i}$; 23. $\sqrt[4]{4-4i}$; 24. $\sqrt[4]{8-8i}$;
25. $\sqrt[3]{1-i^3}$; 26. $\sqrt[3]{i}$.

Bibliography

2. Беклемишев Д. В. Курс аналитической геометрии и линейной алгебры. М.: Наука (издания разных лет).
3. Курош А. Г. Курс высшей алгебры. М.: Наука (издания разных лет).
4. Пискунов Н. С. Дифференциальное и интегральное исчисление. Т.1, 2. М.: Наука (издания разных лет).
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