1. Consider the following theorem:

**Theorem 1.** Let $\beta$ be a nonempty collection of subsets of $X$. If the intersection of any finite number of elements of $\beta$ is always in $\beta$, and if
\[ \bigcup_{B \in \beta} B = X, \]

then $\beta$ is a basis for a topology on $X$.

Use this theorem to do the following. Let $X$ be the real line and let $\beta = \{[a, b) : a < b\}$. Prove that $\beta$ is a base for a topology and that in this topology each member of $\beta$ is both open and closed. (This topology is called the half-open interval topology.)

2. Find a countable basis for the usual topology on $\mathbb{R}$.

[Some remarks on terminology: A topological space with a countable basis is called **second countable**. An example of a space that is not second countable is the real line with the half-open interval topology, defined above (you don’t need to prove this). A related concept is as follows. A space that has a countable dense subset is called **separable**. We’ve already seen an example of this - the real line with the usual topology, which has the rationals as a countable dense set.]

3. Verify the following for arbitrary subsets $A$ and $B$ of a topological space $X$: $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. Show that equality need not hold.

4. Determine the interior, closure, and frontier of each of the following sets.
   
   (a) The plane with both axes removed.
   
   (b) $\mathbb{R}^2 \setminus \{(x, \sin(1/x)) : x > 0\}$

5. Let $X$ be the real line equipped with the finite complement topology. Prove that if $A$ is an infinite set, then every point is a limit point of $A$. In addition, prove that if $A$ is a finite set, then it has no limit points.

6. Prove that $f : X \to Y$ is continuous if and only if $C$ begin closed implies $f^{-1}(C)$ is also closed.

7. Prove that any two open intervals in the real line (with the usual topology) are homeomorphic.

8. Prove that the function defined in lecture is really a homeomorphism between the square and the disk.

9. Let $D$ and $E$ be disks with boundaries $\partial D$ and $\partial E$. Prove that any homeomorphism $h : \partial D \to \partial E$ extends to a homeomorphism from $D$ to $E$. This means there exists a homeomorphism $\tilde{h} : D \to E$ such that $\tilde{h}|_{\partial D} = h$. (You may assume that any homeomorphism from one disk to another maps the boundary of one disk to the boundary of the other.)