1) (From Evans: “Equipartition of Energy”) Let \( u \in C^2(\mathbb{R} \times [0, \infty)) \) solve the initial value problem
\[
\begin{align*}
    u_{tt} - u_{xx} &= 0 \quad \text{in } \mathbb{R} \times (0, \infty) \\
    u &= g, \quad u_t = h \quad \text{on } \mathbb{R} \times \{0\}.
\end{align*}
\]
Suppose \( g \) and \( h \) have compact support. The kinetic and potential energies are, respectively,
\[
\begin{align*}
    k(t) &= \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(x,t) \, dx, \\
    p(t) &= \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(x,t) \, dx.
\end{align*}
\]
Prove that
\[
\begin{align*}
    (a) & \quad k(t) + p(t) \text{ is constant in } t. \\
    (b) & \quad k(t) = p(t) \text{ for all large enough times } t.
\end{align*}
\]

2) (From Evans) Let \( u \) solve
\[
\begin{align*}
    u_{tt} - \Delta u &= 0 \quad \text{in } \mathbb{R}^3 \times (0, \infty) \\
    u &= g, \quad u_t = h \quad \text{on } \mathbb{R}^3 \times \{0\},
\end{align*}
\]
where \( g \) and \( h \) are smooth and have compact support. Show there exists a constant \( C \) such that
\[
|u(x,t)| \leq \frac{C}{t}, \quad \forall \quad x \in \mathbb{R}^3, \quad t > 0.
\]

3) Let \( \Omega \) be a bounded region in \( \mathbb{R}^2 \) with smooth boundary. The motion of a thin, vibrating plate with shape \( \Omega \) and clamped edges is approximated by the equation
\[
\begin{align*}
    u_{tt} &= -\Delta^2 u \quad \text{in } \Omega \times (0, \infty) \\
    u(x,t) = 0, \quad Du(x,t) \cdot \nu &= 0 \quad \text{on } \partial \Omega \times (0, \infty),
\end{align*}
\]
where \( \nu \) is the outward pointing normal vector on \( \partial \Omega \). Show that if we specify initial conditions \( u(x,0) = g(x) \) and \( u_t(x,0) = h(x) \), then this problem has at most one solution. Hint: Try to find a conserved “energy” for this problem.

4) (From Evans) Solve the following equations using the method of characteristics:
   \[
   \begin{align*}
   (a) & \quad x_1 u_{x_1} + x_2 u_{x_2} = 2u, \text{ with boundary data } u(x_1,1) = g(x_1). \\
   (b) & \quad uu_{x_1} + u_{x_2} = 1, \text{ with boundary data } u(x_1,x_1) = x_1/2. \\
   (c) & \quad x_1 u_{x_1} + 2x_2 u_{x_2} = 3u, \text{ with boundary data } u(x_1,x_2,0) = g(x_1,x_2).
   \end{align*}
   \]
Hint: for the last one, something goes wrong. Why?

5) (From Evans) Assume \( F(0) = 0 \), \( u \) is a continuous integral solution of the conservation law
\[
\begin{align*}
    u_t + F(u)_x &= 0 \quad \text{in } \mathbb{R} \times (0, \infty) \\
    u &= g \quad \text{on } \mathbb{R} \times \{0\},
\end{align*}
\]
and \( u \) has compact support in \( \mathbb{R} \times [0, \infty] \). Prove that, for all \( t > 0 \),
\[
\int_{-\infty}^{\infty} u(x,t) \, dx = \int_{-\infty}^{\infty} g(x) \, dx.
\]
6) (From Evans) Explicitly compute the unique entropy solution of

\[
\begin{align*}
    &u_t + \left(\frac{u^2}{2}\right)_x = 0 \quad \text{in } \mathbb{R} \times (0, \infty) \\
    &u = g \quad \text{on } \mathbb{R} \times \{0\},
\end{align*}
\]

where

\[
g(x) = \begin{cases} 
    1 & \text{if } x < -1 \\
    0 & \text{if } -1 < x < 0 \\
    2 & \text{if } 0 < x < 1 \\
    0 & \text{if } 1 < x. 
\end{cases}
\]

Draw a picture illustrating your answer, being sure to illustrate what happens for all times \( t > 0 \).

7) The equation

\[
    u_t + u u_x = \nu u_{xx}
\]

is called Burgers equation with viscosity, where \( \nu > 0 \) is the viscosity parameter. Show that if one defines \( w(x, t) \) via

\[
    -2\nu \log w(x, t) = \int_{-\infty}^{x} u(y, t) dy,
\]

then \( w \) solves the one-dimensional heat equation \( w_t = \nu w_{xx} \). (You can assume \( u \) is smooth with compact support.) This change of variables is known as the Cole-Hopf transformation. Use it to solve Burgers equation with viscosity for \( u(x, 0) = e^{-x^2} \). Plot the solution for several values of \( \nu > 0 \) that approach the limit \( \nu = 0 \). Describe how shocks are forming. What effect does the viscosity, i.e., the presence of the Laplacian \( u_{xx} \), have? Does this make sense, based on what you know about the properties of solutions of the heat equation and Poisson’s equation?