1. Inner product (also known as dot product):

In $\mathbb{R}^n$:
Orthogonal vectors

Inner product of \( \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \) and \( \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \) is

\[
\sum_{i=1}^{n} v_i w_i
\]

Norm of (real) vector:

\[
\sqrt{v_1^2 + v_2^2 + \ldots + v_n^2} = \sqrt{\sum_{i=1}^{n} v_i^2} = \| \mathbf{v} \| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}.
\]
2. Inner product, geometric:

3 D:

\[ \mathbf{v} \cdot \mathbf{w} = \| \mathbf{v} \| \| \mathbf{w} \| \cos \theta \]

where \( \theta \) = angle between \( \mathbf{v} \) and \( \mathbf{w} \).
Geometry:

\[ \mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \theta = \pi/4, \]

Inner product is: \[ \mathbf{v} \cdot \mathbf{w} = 1 \cdot 1 + 0 \cdot 1 = 1 \]
Inner product

On other hand can use:

\[ ||v|| \cdot ||w|| \cos \theta = 1 \cdot \sqrt{2} \cdot \frac{1}{\sqrt{2}} = 1 = v \cdot w \]

to get same value.
Inner product

3. Properties of IP:

**Theorem 1:** For \( u, v \in \mathbb{R}^n \),

(a) \[ \langle u, u \rangle \geq 0 \] for all \( u \); \( \langle u, u \rangle = 0 \) iff \( u = 0 \).

(b) \[ \langle u, v \rangle = \langle v, u \rangle \]

(or \( \langle u, v \rangle = \langle v, u \rangle \) if \( u, v \) are complex)

(c) \[ \langle [u + v], w \rangle = \langle u, w \rangle + \langle v, w \rangle \]

(d) \[ \langle u, cv \rangle = c\langle u, v \rangle \] if \( c \) is a real scalar.

**Exercise:** Verify these properties for \( v, w \in \mathbb{R}^n \).

**Def. 6.** \( v \) is a *unit vector* if has length 1
Inner product

e.g., \( \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \) = unit vector (after normalization).

4. Inner Product, general:

Can we define IP on abstract vector spaces?

**Definition 1:** \( V \) = vector space. An *inner product* on \( V \) is any assignment of a numerical value to \( \langle u, v \rangle \) which satisfies properties (a) to (d) of the above theorem.

Any vector space with IP defined is an *inner product space*. 
General inner products

**Ex 1:** Standard IP on \( \mathbb{R}^3 \);

\[
\mathbf{u} \cdot \mathbf{v} = \sum_i u_i v_i
\]

[already know this satisfies (a) to (d)].

**Ex 2:** Recall

\[
C[-\pi, \pi] = \text{continuous functions on } [-\pi, \pi]
\]

are vector space

[\text{note } C[-\pi, \pi] \text{ has a basis } \{1, \sin nx, \cos nx\}_{n=1}^{\infty}; \text{ infinite dimensional}]}
General inner products

Define inner product of functions:

\[ f \cdot g = \langle f, g \rangle = \int_{-\pi}^{\pi} f(x) \, g(x) \, dx. \]

[satisfies (a)-(d); check this]

Property (a) \((f, f) \geq 0; (f, f) = 0 \iff f = 0.\)
General inner products

**Pf:** We have

\[(f, f) = \int_0^1 f^2(x) \, dx \geq 0\]

since \( f^2 \) is non-negative. Also, if \((f, f) = 0\) then \( \int_0^1 f^2 \, dx = 0 \). From calculus, if a function is nonnegative and its integral is 0, then \( f(x) = 0 \), as desired. Also, if \( f = 0 \) then clearly \((f, f) = 0\). □

Other facts proved from the definitions the same way.
General inner products

3. Schwarz inequality and triangle inequality:

Theorem 2 (Cauchy-Schwarz inequality):

\[ |(u, v)| \leq \| u \| \| v \| \]

for any two vectors in a vector space with an inner product.

Proof: Standard in linear algebra.
General inner products

Ex 3: \[ u = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}; \quad v = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}; \quad \text{then} \quad u \cdot v = -2, \text{ while}
\]

\[ \|u\| \|v\| = \sqrt{6} \sqrt{9} = 3\sqrt{6} \geq | -2 |,
\]

as desired.
General inner products

**Theorem 3 (Triangle inequality):** For any two vectors $u$ and $v$, we have $\|u + v\| \leq \|u\| + \|v\|$.

**Proof:** standard in linear algebra.
4. **Orthogonal vectors:**

Three dimensions: vectors \( \mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \) and \( \mathbf{w} = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} \) have zero dot product.
Thus we know that \[ \| \mathbf{v} \| \| \mathbf{w} \| \cos \theta = 0 \Rightarrow \theta = \pi/2. \]

Thus the vectors are perpendicular or *orthogonal.*
Orthogonality

More generally, what happens with higher dimensional vectors?

Note \begin{bmatrix} 2 \\ -3 \\ 1 \\ -1 \end{bmatrix} and \begin{bmatrix} -2 \\ -1 \\ 0 \\ -1 \end{bmatrix} also have zero dot product.

Perpendicular
Orthogonality

**Def 2:** We define two vectors \( \mathbf{v} \) and \( \mathbf{w} \) to be *perpendicular* or *orthogonal* if

\[
\mathbf{v} \cdot \mathbf{w} = 0.
\]

[Note this implies angle between them is \( \theta = \pi/2 \)]

**Def 3:** A collection \( S = \{v_1, v_2, \ldots, v_n\} \) of vectors is *orthogonal* if each pair of the vectors \( v_i \) is orthogonal. A collection is *orthonormal* if they are orthogonal and all are unit vectors.
Ex 4:

\[
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix},
\begin{bmatrix}
1 \\
1 \\
-2
\end{bmatrix},
\begin{bmatrix}
-1 \\
1 \\
0
\end{bmatrix}
\]

are orthogonal, not orthonormal.
Orthogonality

\[
\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\]

are orthonormal.

[just check dot products and lengths]
Orthogonality

fig 5
Orthonormal bases

3. Advantage of orthonormal bases:
Given an orthonormal basis \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \), how do we express a vector \( \mathbf{w} \) in terms of them?

Ex 5: Assume that

\[
\mathbf{v}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad \mathbf{v}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}
\]

assume \( \mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \).
Orthonormal bases

Notice that:

\[ \mathbf{w} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3. \]

Then:

\[ (\mathbf{w}, \mathbf{v}_1) = c_1 = \frac{4}{\sqrt{3}} \]
Similarly

\[(w, v_2) = c_2 = \frac{1}{\sqrt{6}}\]

\[(w, v_3) = c_3 = -\frac{1}{\sqrt{2}}.\]

Thus expansion is easy to get with orthonormal bases!

[Orthonormal bases make such computations easy].