Measurability and Hilbert Spaces

1. Measurable functions and integrals

Let \( C \) be the set of continuous functions on \( \mathbb{R} \). Let \( M \) be the set of measurable functions:

**Def:** The set \( M \) of measurable functions on \( \mathbb{R} \) (or an interval of \( \mathbb{R} \)) is the set of pointwise limits of continuous functions, i.e.
Measureable functions

\[ M = \{ f(x) : f(x) = \lim_{n \to \infty} f_n(x), \text{ where } f_n(x) \text{ is continuous} \} \]

Note discontinuous functions can be measurable: for example, 
\[ f(x) = \chi_{[0,1]}(x) \] is a measurable function.
Measureable functions
Measureable functions

Fig. 1: the function $f(x)$ as a limit of continuous functions.

In fact, lots of functions (even discontinuous ones) can be viewed as limits of continuous functions.

Note: ordinary notion of integral is difficult to use for functions as complicated as measurable functions.

To integrate measurable functions (Lebesgue integral):

Theorem: Given a non-negative measurable function $f : \mathbb{R}^p \to \mathbb{R}$, there is always an increasing sequence $\{f_n(x)\}_{n=1}^{\infty}$ of continuous functions (i.e. with the property that $f_{n+1}(x) \geq f_n(x)$ for all $x$) which converges to $f(x)$. 
Integrals of measurable functions

**Def.** If $f(x) \geq 0$ is a positive measurable function, define

$$
\int_{\mathbb{R}^p} f(x) \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^p} f_n(x) \, dx,
$$

where $f_n(x)$ is any increasing sequence of continuous functions which converges to $f$.

[note we know the value of the integrals of the continuous functions $f_n(x)$ - they are ordinary Riemann integrals on $\mathbb{R}^p$]
Integrals of measurable functions

6a: sequence of continuous functions $f_n(x)$ increasing to
Integrals of measurable functions

\[ f(x) \]

**Def:** To find the integral of a negative measurable function \( f \), we just compute the integral of \( -f \) (which is positive), and put a minus sign in front of it. Since every function \( f \) is the sum of a positive plus a negative function

\[ f = f_1 + f_2, \]

the integral of \( f \) is defined as

\[ \int_{-\infty}^{\infty} f \, dx = \int_{-\infty}^{\infty} f_1 \, dx + \int_{-\infty}^{\infty} f_2 \, dx. \]
Integrals of measurable functions

[Thus we now know how to define the integral of an arbitrary function]

Ex: if \( f(x) \) looks like:
Integrals of measurable functions

fig 2: $f(x)$ has positive and negative part
Integrals of measurable functions

Then integral of $f(x)$ is integral of a positive plus a negative function:

\[ \int_{a}^{b} f(x) \, dx = \int_{a}^{b} g(x) \, dx + \int_{a}^{b} h(x) \, dx \]

fig 8: now sum the areas between $f_1$ (or $f_2$) and the x axis
Integrals of measurable functions

Note we can show pretty easily all the properties of integrals we are used to also hold for this more general Lebesgue integral. For example, we still have

\[ \int (f + g) \, dx = \int f \, dx + \int g \, dx, \text{ etc.} \]

[May assume this fact.]
Hilbert spaces of functions

2. New Hilbert spaces:

Consider the space

\[ H = L^2[-\pi, \pi] = \{ \text{measurable real functions } f(x) \text{ on } [-\pi, \pi] \text{ with } \int_{-\pi}^{\pi} f^2(x) \, dx < \infty \}. \]

Can show that if \( f, g \in H \) then \( f + g \) and \( cf \) are in \( H \) if \( c \) is a constant (exercise). More generally \( H \) is a vector space.

Further, we can define an inner product on \( H \):

\[ (f, g) = \int_{-\pi}^{\pi} f(x) \, g(x) \, dx. \]
Hilbert spaces of functions

This satisfies conditions (1) - (4) of an inner product.

Can also show that $H$ is complete (i.e., every Cauchy sequence $\{f_n\}$ converges to a function $f$ in $H$).

Thus $H$ is a Hilbert space.

Note: we always consider two measurable functions the same if they differ just at a finite number of points.
Hilbert spaces of functions

fig 7a two functions $f_1$ and $f_2$ which differ at a finite collection of points.
Hilbert spaces of functions

Can show: such functions $f_1$ and $f_2$ have the same integral [certainly area is unchanged]; equivalently,

$$\int |f_1 - f_2| \, dx = 0$$

Def: More generally we will consider two functions to be the same or equivalent if (1) holds
Ex: Let $H = L^2[-\pi, \pi]$ . Consider the set of vectors

$$B = \{ \sin nx | n = 1, 2, \ldots \} \quad \text{together with} \quad \{ \cos nx | n = 0, 1, 2, \ldots \}$$

$$= \{ 1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots \}$$

We will show this is an orthogonal set. First: show that 1 is orthogonal to all other vectors:

$$(1, \cos nx) = \int_{-\pi}^{\pi} \cos nx \, dx = 0 \quad \forall \ n = 1, 2, \ldots$$
Function space basis expansions

\[(1, \sin nx) = \int_{-\pi}^{\pi} \sin nx \, dx = 0 \quad \forall \ n = 1, 2, \ldots\]

Now show say that \(\cos 5x\) is orthogonal to all other vectors:

\[(\cos 5x, \sin nx) = \int_{-\pi}^{\pi} \cos 5x \sin nx \, dx = 0 \quad \forall \quad n = 1, 2, \ldots\]

Using:

\[
\cos a \cos b = \frac{1}{2} [\cos (a + b) + \cos (a - b)]
\]

and
Function space basis expansions

\[
\sin a \cos b = \frac{1}{2} \left[ \sin (a + b) + \sin (a - b) \right]
\]

\[
\sin a \sin b = -\frac{1}{2} \left[ \cos (a + b) - \cos (a - b) \right].
\]

[Similarly for any other \( \cos mx \).]

\[
(\cos 5x, \cos nx) = \int_{-\pi}^{\pi} \cos 5x \cos nx \, dx = 0 \quad \forall \quad n \neq 5
\]

Can similarly show that \( \sin mx \) is also orthogonal to all other vectors.
Function space basis expansions

Thus these vectors form a orthogonal set of vectors. Are they orthonormal?

\[ \| \cos nx \|^2 = (\cos nx, \cos nx) = \int_{-\pi}^{\pi} \cos^2 nx \, dx \]

\[ = \int_{-\pi}^{\pi} \frac{1 + \cos 2nx}{2} \, dx \]

\[ = \pi \]
Function space basis expansions

Thus:

$$\|\cos nx\| = \sqrt{\pi}.$$ 

Thus

$$\frac{1}{\sqrt{\pi}} \cos nx \text{ has length 1.}$$

Similarly,

$$\frac{1}{\sqrt{\pi}} \sin nx \text{ has length 1}.$$ 

And:

$$\frac{1}{\sqrt{2\pi}} \cdot 1 \text{ has length 1.}$$
Function space basis expansions

Thus:
\[ \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos x, \frac{1}{\sqrt{\pi}} \sin x, \frac{1}{\sqrt{\pi}} \cos 2x, \frac{1}{\sqrt{\pi}} \sin 2x, \frac{1}{\sqrt{\pi}} \cos 3x, \frac{1}{\sqrt{\pi}} \sin 3x, \ldots \right\} \]

\[ = \{v_1, v_2, v_3, \ldots \} \]

Are an orthonormal (and hence lin ind) set for the space of cont. functions.

Can show: they are a basis. So any vector \( f(x) \) can be written in the form:
Function space basis expansions

\[ f(x) = c_1 v_1 + c_2 v_2 + \ldots \]

\[ = c_1 \frac{1}{\sqrt{2\pi}} + c_2 \frac{1}{\sqrt{\pi}} \cos x + c_3 \frac{1}{\sqrt{\pi}} \sin x + c_4 \frac{1}{\sqrt{\pi}} \cos 2x\]

\[ + c_5 \frac{1}{\sqrt{\pi}} \sin 2x + \ldots \]

\[ = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots \]
Function space basis expansions

[Fourier series of a function]

Notice that

\[ c_4 = \left( f(x), \frac{1}{\sqrt{\pi}} \cos 2x \right) = \int_{-\pi}^{\pi} f(x) \frac{1}{\sqrt{\pi}} \cos 2x \, dx \]

\[ = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) \cos 2x \, dx \]

\[ \Rightarrow a_2 = \frac{c_4}{\sqrt{\pi}} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos 2x \, dx \]

Generally:
Function space basis expansions

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \]

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx. \]

[Now: no need to do advanced calculus for theory of Fourier series!]

Function space basis expansions

Ex: \[ f(x) = 2x \]
Function space basis expansions

fig 6

\[ 2x = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots \]

\[
= \frac{1}{\pi} \int_{-\pi}^{\pi} 2x \sin 5x \, dx = \frac{2}{\pi} \left\{ - \frac{\cos 5x}{5} \right\}_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos 5x}{5} \, dx
\]

\[ = \frac{2}{\pi} \left\{ \frac{2\pi}{5} \right\} = \frac{4}{5} \]
Function space basis expansions

\[ b_6 = -\frac{4}{6} \]

Generally:

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} 2x \cos nx \, dx = \begin{cases} -\frac{4}{n} & \text{if } n \text{ even} \\ \frac{4}{n} & \text{if } n \text{ odd} \end{cases} \]

Can show \[ a_n = 0. \]
Function space basis expansions

Thus

$$2x = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots$$

$$= 4 \left[ 1 \cdot \sin x - \frac{1}{2} \cdot \sin 2x + \frac{1}{3} \cdot \sin 3x + \ldots \right]$$

[can draw pictures of first three terms (see earlier); all divided by 2 for the function x]