5.4. (a) Recall natural splines are piecewise cubic on \([a, b]\) with knots at \(\{\xi_1, ..., \xi_K\}\) which are restricted as linear in \([a, \xi_1]\) and in \([\xi_K, b]\). Show linearity in the first interval is equivalent to \(\beta_2 = \beta_3 = 0\). In \([\xi_K, b]\) how does \(f(X)\) simplify? Use this fact to get the remaining conditions.

(b) For the basis functions in (5.4), you can prove they are a basis for natural splines by showing (i) they are in the space, (ii) there are \(N\) of them, and (iii) they are linearly independent. Why are these sufficient? For (i) recall the conditions in (a) above. For linear independence, why is it enough to show that each function in list (5.4) is linearly independent of the previous ones?

To show that each \(N_k\) is linearly independent of the previous \(N_1, ..., N_{k-1}\), note \(N_1 = 1, N_2 = x\) while

\[
N_3(x) = d_1(x) - d_{K-1}(x) = \frac{(x - \xi_1)\beta_3 - (x - \xi_3)\beta_3}{\xi_K - \xi_1} - \frac{(x - \xi_{K-1})\beta_3 - (x - \xi_K)\beta_3}{\xi_K - \xi_{K-1}},
\]

so \(N_3\) has the property that it has a discontinuity in its third derivative at \(\xi_1\), because of the term \((x - \xi_1)^3\). Since \(N_1\) and \(N_2\) do not have such discontinuities, it follows that \(N_1\) cannot be a linear combination of \(N_0, N_1\). Similarly, \(N_4\) has a third derivative discontinuity at \(\xi_2\), and so cannot be a combination of the previous functions, none of which have derivative discontinuities at \(\xi_4\). Continue this way, until noting that \(N_{K-1} = d_{K-3} - d_{K-1}\) has a third derivative discontinuity at \(\xi_{K-3}\), while none of the previous \(N_k\) do, and \(N_K\) has a third derivative discontinuity at \(\xi_{K-2}\), while none of the previous ones do. Thus conclude each function is linearly independent of the previous ones. How does this complete the proof?

5.7. (a) You can show by breaking up the left side that

\[
\int_a^b g''(x)h''(x)dx =
\]

\[
= \sum_{j=1}^{K-1} \int_{x_{j+1}}^{x_j} g''(x)h'(x)dx + \sum_{j=1}^{K-1} \{g''(x_{j+1})h'(x_{j+1}) - g''(x_j)h'(x_j)\}.
\]

(why is the first term \(\int_a^{x_1} g''(x)h'(x)dx = 0\)?) Show after cancellations the second sum equals

\[
g''(x_K)h'(x_K) - g''(x_1)h'(x_1) = 0
\]

(what must \(g''(x_1)\) be?). You can IBP again obtaining
\[
= - \sum_{j=1}^{K-1} g'''(x) h(x)_{x_j}^{x_{j+1}} \tag{1}
\]

(why is \( g''' = 0 \)?) You need to be careful defining the above sum — remember that \( g'''(x_j^+) \neq g'''(x_j^-) \), where + means right hand limit. Note \( g'''(x_{j+1}) = g'''(x_j^+) \) (why?), to obtain from (1)

\[
- \sum_{j=1}^{N-1} g'''(x_j^+) [h(x_{j+1}) - h(x_j)] = 0.
\]

Why is \( h(x_j) = 0 \)?

(b) You can try using

\[
\int_a^b \tilde{g}''(x)^2 \, dx = \int_a^b [\tilde{g}''(x) - g''(x) + g''(x)]^2 \, dx.
\]

(c) If \( g(x) \) is differentiable, consider the Lagrangian penalty

\[
\mathcal{L}(g) = \sum_{j=1}^{N} (y_i - g(x_i))^2 + \lambda \int_a^b g''(x)^2 \, dx.
\]

Show that the cubic spline solution \( g(x) \) above makes both the first and the second terms the smallest possible.