Suggestions for Problem Set 5

1. (a) Recall that a function $f$ is defined to be piecewise linear on a region $R$ in $\mathbb{R}^p$ if $R$ consists of a finite disjoint collection $R = \bigcup_i R_i$ of sub-regions $R_i$ such restricted to any $R_i$, $f$ is a linear function of the form $\beta^T x + b$, with $\beta \in \mathbb{R}^p$ a constant vector.

To prove this it may be worthwhile to first prove:

**Lemma:** The maximum $m(x) \equiv \max(f_1(x), f_2(x))$ of two continuous functions on a region $R \subset \mathbb{R}^p$ is continuous.

To prove this, you can show that at any fixed $y \in \mathbb{R}^p$ at which $f_1(y) > f_2(y)$, by continuity it follows $f_1(x) > f_2(x)$ for $x$ sufficiently close to $y$, and so clearly $m(x) \equiv f_1(x)$ near $y$, so $m$ is continuous at the point $y$ (why?). Similarly show the same holds at any $y$ where $f_1(y) < f_2(y)$.

Finally, if $f_1(y) = f_2(y)$ at some point $y$, then you can use the $\epsilon$-$\delta$ definition of continuity to show $m$ is continuous at $y$. That is, show for any $\epsilon > 0$ there is a $\delta > 0$ such that if $|x - y| < \delta$ then $|m(x) - m(y)| < \epsilon$. Do this by first finding a $\delta_1$ such that if $|x - y| < \delta_1$ then $|f_1(x) - f_1(y)| < \epsilon$, and then a $\delta_2$ such that if $|x - y| < \delta_2$ then $|f_2(x) - f_2(y)| < \epsilon$ (why do these exist?).

Then show that if $|x - y| < \min(\delta_1, \delta_2)$ then

$$|m(x) - m(y)| \leq \max(|f_1(x) - f_1(y)|, |f_2(x) - f_2(y)|) < \epsilon.$$

Thus you should have found the $\delta$ you needed for $m$, namely $\delta = \min(\delta_1, \delta_2)$. Now why is $m$ continuous?

**Lemma:** The maximum of two linear functions $f_1$ and $f_2$ on a region $R$ is piecewise linear and continuous.

To prove this, note that $R = R_1 \cup R_2$, where $R_1 = \{x : f_1(x) \geq f_2(x)\}$, and $R_2 = R \sim R_1$. Show that $m(x) = f_1(x)$ in $R_1$ and $m(x) = f_2(x)$ in $R_2$, and hence that $m(x)$ is piecewise linear.

To prove the theorem, for $f_1, f_2$ piecewise linear you can consider the collection of regions $R_i$ with $\bigcup_i R_i = R$ such that $f_1$ is linear on each $R_i$ (why does this exist?), and the collection $S_j$ with $\bigcup_j S_j$ such that $f_2$ is linear on each $S_j$, so that each $f_i$ is piecewise linear. Why is $m(x)$ linear on each domain $S_j \cap R_i$? How does that prove the result?

(b) To show $f_1 + f_2$ is continuous piecewise linear, try a similar argument

(c) To show $f_2(f_1(x))$ is continuous linear, this time consider the collection $R_i$ of regions on which $f_1$ is linear, and the collection $S_j$ of regions on which $f_2$ is linear. Note
that here each $S_j$ is just a line segment, since we assume $f_1$ maps into $\mathbb{R}$, so the domain of $f_2$ is $\mathbb{R}$ (why?).

Try to show that for each $S_j \subset \mathbb{R}$, the set $f_2^{-1}(S_j) = \{ \mathbf{x} : f_2(\mathbf{x}) \in S_j \}$ is a region $T_j$ in $\mathbb{R}^p$. Show then that on the each nonempty set $R_i \cap T_j$, the function $f_2(f_1(\mathbf{x}))$ is linear. How does that prove piecewise linearity?

Finally, you may assume the result from calculus which says that the composition of two continuous functions is continuous, but please state this (known) fact precisely.

2. For a simple introduction to the use of Tensorflow, take a look at https://www.tensorflow.org/tutorials?authuser=1