Lectures 10, 11

The material in this problem set consists of elementary notions in linear algebra and functional analysis that we will need for the discussion of reproducing kernel Hilbert spaces. The basics of this topic are not handled thoroughly in the textbook, but are covered fully by the class discussion and these problems.

1. (a) Prove the functions $v_1 = \sin x + 2x^2$; $v_2 = 2\sin x - e^x$; $v_3 = e^x$ are linearly independent as elements of the vector space of all continuous functions on the real line $\mathbb{R}$.

(b) Prove this set of vectors is a basis for the vector space $V$ which consists of all linear combinations of the functions $\sin x$, $x^2$, and $e^x$ defined for real $x$.

2. For two continuous real-valued functions $f$ and $g$ defined on $[0,1]$, show that the definition $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$ satisfies all the properties of an inner product.

3. Let $S = \{v_1, \ldots, v_n\}$ be an orthonormal basis for the vector space $V$. Show that any vector $w \in V$ can be written

$$w = c_1 v_1 + \ldots + c_n v_n,$$

with $c_i = \langle v_i, w \rangle$.

4. In the construction of the Lebesgue integral in class, we have assumed we only know how to calculate the integrals of continuous functions using the Riemann integral (i.e., what you learned in calculus I). The Lebesgue integral extends the notion of integral to a much larger class of functions.

As an exercise in how this integral works, compute the Lebesgue integral

$$\int_{-\infty}^{\infty} I_{[0,1]}(x)dx$$

using the definition of the integral as a limit of integrals of continuous functions introduced in class. Here $I_{[0,1]}(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$.

Show that the integral you obtain is just the usual area under the function $I_{[0,1]}(x)$.

5. Completeness: Show that the vector space $H = \mathbb{R}^3 = \{v = (\nu_1, \nu_2, \nu_3) | \nu_i \in \mathbb{R}\}$ with inner product defined by $(u, v) = \sum_{i=1}^{3} u_i \nu_i$ is complete.

6. Fourier series: Find the Fourier series (on the interval $(-\pi, \pi)$) of the function $f(x)$.

$$f(x) = \begin{cases} 1 & \text{if } -\pi \leq x < 0 \\ -1 & \text{if } 0 \leq x \leq \pi \end{cases}.$$
7. **The square integrable functions form a Hilbert space:** Here we will try to establish that if $A$ is a compact (i.e., closed and bounded) set in $\mathbb{R}^p$, then $L^2(A) = \{ \text{measurable functions } f(x) \text{ defined on } x \in A \text{ such that } \int_A f^2(x)dx < \infty \}$ is a Hilbert space. Recall again a measurable function $f(x)$ is a function which is a limit of some sequence of continuous functions $f_n(x)$, where $x = (x_1, x_2, \ldots, x_n)$.

(a) Show that the sum of two measurable functions $f(x)$ and $g(x)$ is measurable.

(b) Let $A$ be a compact set. Show that $H = L^2(A) = \{ \text{measurable functions } f(x) \text{ on } A \text{ with } \int_A f^2(x)dx < \infty \}$ is a vector space.

(c) Show that if $A$ is a compact set in $\mathbb{R}^d$, the inner product of two functions $f$ and $g$ on $A$, defined by

$$\langle f, g \rangle = \int_A f(x)g(x)dx,$$

satisfies the four properties of an inner product. You may assume the basic fact of measure theory that a non-negative measurable function $f$ which satisfies $\int_A f(x)dx = 0$ must be the zero function, and may use the fact that the basic properties of integrals hold for integration of measurable functions (e.g., integral of a sum is a sum of integrals, etc.).

This will show that $L^2(A)$ is an inner product space. The fact that it is complete (and thus a Hilbert space) can also be shown, though we will not do this here.

8. A reproducing kernel Hilbert space $\mathcal{H}$ has a reproducing kernel $K(\cdot, \cdot)$. For $g(x) = \sum_{i=1}^n c_i K(x, x_i)$ and $h(x) = \sum_{i=1}^n d_i K(x, x_i)$, find $\|g\|_{\mathcal{H}}$, $\|h\|_{\mathcal{H}}$ and $\langle g, h \rangle_{\mathcal{H}}$. 