Lectures 11, 12, 13

Because of the snowstorm and snow day on March 13, this problem set now covers Lectures 11, 12 and 13, and will be due a week later, Thursday March 22. There are no additional problems.

This material largely covers the extended discussion we have had involving functional analysis and reproducing kernel Hilbert spaces. The problems complete some of this discussion.

1. Fourier series: Find the Fourier series (on the interval \((-\pi, \pi, \)) of the function \(f(x) = \begin{cases} \frac{1}{2} & \text{if } -\pi \leq x < 0 \\ -1 & \text{if } 0 \leq x \leq \pi \end{cases} \).

2. In the construction of the Lebesgue integral in class, we have assumed we only know how to calculate the integrals of continuous functions using the Riemann integral (i.e., what you learned in calculus 1). The Lebesgue integral extends the notion of integral to a much larger class of functions.

As an exercise in how this integral works, compute the Lebesgue integral \(\int_{-\infty}^{\infty} I_{[0,1]}(x) \, dx \) using the definition of the integral as a limit of integrals of continuous functions introduced in class. Here \(I_{[0,1]}(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \).

Show that the integral you obtain is just the usual area under the function \(I_{[0,1]}(x)\).

3. The square integrable functions form a Hilbert space: Here we will try to establish that if \(A\) is a compact (i.e., closed and bounded) set in \(\mathbb{R}^p\), then \(L^2(A) = \{\text{measurable functions } f(x) \text{ defined on } x \in A \text{ such that } \int_A f^2(x) \, dx < \infty\}\) is a Hilbert space. Recall again a measurable function \(f(x)\) is a function which is a limit of some sequence of continuous functions \(f_n(x)\), where \(x = (x_1, x_2, \ldots, x_p)\).

(a) Show that the sum of two measurable functions \(f(x)\) and \(g(x)\) is measurable.
(b) Let \(A\) be a compact set. Show that \(H = L^2(A) = \{\text{measurable functions } f(x) \text{ on } A \text{ with } \int_A f^2(x) \, dx < \infty\}\) is a vector space.
(c) Show that if \(A\) is a compact set in \(\mathbb{R}^p\), the inner product of two functions \(f\) and \(g\) on \(A\), defined by
\[ \langle f, g \rangle = \int_A f(x)g(x)\,dx, \]
satisfies the four properties of an inner product. You may assume the basic fact of measure theory that a non-negative measurable function \( f \) which satisfies \( \int_A f(x)\,dx = 0 \) must be the zero function, and may use the fact that the basic properties of integrals hold for integration of measurable functions (e.g., integral of a sum is a sum of integrals, etc.).

This will show that \( L^2(A) \) is an inner product space. The fact that it is complete (and thus a Hilbert space) can also be shown, though we will not do this here.

4. A reproducing kernel Hilbert space \( \mathcal{H} \) has a reproducing kernel \( K(\cdot, \cdot) \). For \( g(x) = \sum_{i=1}^n c_i K(x, x_i) \) and \( h(x) = \sum_{i=1}^n d_i K(x, x_i) \), find \( \|g\|_\mathcal{H}, \|h\|_\mathcal{H} \) and \( \langle g, h \rangle_\mathcal{H} \).

5. Hastie, problem 5.15

6. (More on RHKS) Consider the Hilbert space \( L^2(F) \) of square integrable functions where \( F \subset \mathbb{R}^d \) is compact (i.e. closed and bounded), together with an orthonormal basis \( \{\phi_k(x)\}_{k=1}^\infty \) of uniformly bounded functions (i.e. \( |\phi_k(x)| \leq M \) for some fixed \( M \) and for all \( k, x \)). Consider now the sub-space \( \mathcal{H} \) consisting of functions \( f(x) = \sum_{k=1}^\infty a_k \phi_k(x) \in L^2(F) \), with inner product

\[ \langle f, g \rangle_\mathcal{H} = \sum_{k=1}^\infty a_k b_k \gamma_k. \]

A function \( f \in \mathcal{H} \) iff \( \|f\|_\mathcal{H}^2 = \langle f, f \rangle_\mathcal{H} < \infty \) for \( f \) as above and \( g = \sum_{k=1}^\infty b_k \phi_k(x) \), with \( \gamma_k > 0 \) for all \( n \).

(a) Prove that \( \mathcal{H} \) is a Hilbert space (proving that it is complete is optional).
(b) Find conditions under which \( \mathcal{H} \) is an RKHS. You can answer this by finding a simple condition on the \( \{\gamma_k\} \) which would guarantee \( \mathcal{H} \) is an RKHS in the sense of our original definition.
(c) Find the reproducing kernel \( K(x, y) \) in this case. Show that \( K \) has the reproducing property \( \langle f(\cdot)K(\cdot, x) \rangle_\mathcal{H} = f(x) \) for all \( f \in \mathcal{H} \).