5. (Hastie 5.15)  
(a) Keep in mind what the inner product on this space is: if\( f(\mathbf{x}), g(\mathbf{x}) \in \mathcal{H} \) with 
\[
f(\mathbf{x}) = \sum_{i=1}^{\infty} a_i \phi_i(\mathbf{x}), \quad g(\mathbf{x}) = \sum_{i=1}^{\infty} b_i \phi_i(\mathbf{x}), \quad \text{then} \quad \langle f, g \rangle = \sum_{i=1}^{\infty} \frac{a_i b_i}{\gamma_i}.
\]
You can write 
\[
K(\cdot, \mathbf{x}_i) = \sum_{i=1}^{\infty} \{\gamma_k \phi_k(\mathbf{x}_i)\} \phi_k(\cdot) ; \quad f(\cdot) = \sum_{k=1}^{\infty} a_k \phi_k(\cdot);
\]
how does it follow that (a) holds just using definitions?  
(b) Use a similar representation to that in (a)  
(d) When is \( \rho(\cdot) \in \mathcal{H} \) orthogonal (in \( \mathcal{H} \)) to \( K(\cdot, \mathbf{x}_i) \) for fixed \( \mathbf{x}_i \) (i.e. they have dot product 0 in the active variable)? Show \( \rho(\mathbf{x}_i) = 0 \). How does this affect the Lagrangian \( \sum_{i=1}^{N} L(\hat{f}(\mathbf{x}_i), y_i) \)? What happens to \( ||\hat{f}||^2_{\mathcal{H}} \) ?  

6. (More on RHKS).  
Note that we have two Hilbert spaces here. \( L^2(F) \) is the Hilbert space of all square integrable functions, i.e. such that \( \int_{F} f^2(x)dx < \infty \). The inner product on \( L^2(F) \) is defined as \( \langle f(x), g(x) \rangle_{L^2(F)} = \int_{F} f(x)g(x)dx \). Note that the inner product in the smaller subspace \( \mathcal{H} \subset L^2(F) \) is \( \langle f(x), g(x) \rangle_{\mathcal{H}} \). Also, we assume function \( f \in \mathcal{H} \) iff \( ||f||^2_{\mathcal{H}} = \langle f, f \rangle < \infty \). Note that in general the values of \( \gamma_k \) are assumed to go to 0, so that convergence of the norm \( ||g||^2_{\mathcal{H}} = \sum_{k} a_k^2 / \gamma_k \) of the function \( g(x) = \sum_{k} a_k \phi_k(x) \) requires the coefficients \( a_k \) to go to 0 faster than the condition of finiteness of \( ||g||^2_{L^2(F)} = \sum_{k} a_k^2 \). Consider the example where the functions \( \phi_k \) are just the Fourier series functions \( \sin kx \) and \( \cos kx \) on the interval \( F = [-\pi, \pi] \subset \mathbb{R} \).  

If we require the coefficients \( a_k \) and \( b_k \) to go to 0 rapidly in the Fourier series \( g(x) = \sum_{k} a_k \cos kx + b_k \sin kx \), this (as we have shown) will make \( g \) smoother. Thus the condition that \( g \in \mathcal{H} \), i.e., that \( ||g||^2_{\mathcal{H}} = \sum_{k} a_k^2 / \gamma_k < \infty \), is a smoothing condition and essentially requires that our Hilbert space \( \mathcal{H} \) be a space of smooth functions on the same domain \( F \) as \( L^2(F) \).  

Note more generally that the requirement in the Lagrangian that \( ||g||^2_{\mathcal{H}} \) be small is a requirement that that \( a_k \) go to 0 faster as \( k \to \infty \), the bigger the \( 1/\gamma_k \) are. Again this becomes a smoothness requirement, since the 'unsmooth' parts of \( g \) are the components with high \( k \).
(a) Again show it is closed under addition, etc., and that the inner product defined satisfies the right properties.
(b) For \( \mathcal{H} \) to be an RKHS, the linear functional \( l(f) = f(x) \) must be bounded for any fixed \( x \) (see notes). Show for fixed \( x \) we need \( |f(x)| \leq C||f||_{\mathcal{H}} \) (for all \( f \)). Show it suffices that
\[
\sum_k \gamma_k \phi(x_k)^2 < A < \infty \quad \text{for all } x_1 \in F \text{ with some constant } A. \quad (1)
\]

How can you simplify condition (1)? Note that if \( \sum \gamma_k < \infty \) then
\[
\sum_k \gamma_k \phi(x_1)^2 \leq M^2 \sum_k \gamma_k.
\]

Note also the Schwarz inequality
\[
\left| \sum_k a_k b_k \right| \leq \sqrt{\sum_k a_k^2} \sqrt{\sum_k b_k^2}.
\]

Show that then that if \( f(x) = \sum_k c_k \phi_k(x) \), then
\[
|f(x)| = \left| \sum_k c_k \phi_k(x) \right| = \left| \sum_k \frac{c_k}{\sqrt{\gamma_k}} (\sqrt{\gamma_k} \phi_k(x)) \right| \leq \left( \sum_k \frac{c_k^2}{\gamma_k} \right) \left( \sum_k \gamma_k \phi_k^2(x) \right).
\]

(c) How about \( \sum \phi_k(x) \phi_k(y) \) - does this work? Explain carefully. Show that if \( f = \sum c_k \phi_k(x) \in \mathcal{H} \), then
\[
\langle K(x, \cdot), f(\cdot) \rangle_{\mathcal{H}} = \left\langle \sum_k \gamma_k \phi_k(x) \phi_k(\cdot), \sum l \phi_l(\cdot) \right\rangle_{\mathcal{H}} = f(x).
\]