1. Meyer wavelets: another example -

Scaling function:

\[
\hat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}} \begin{cases} 
1 & \text{if } |\omega| \leq 2\pi/3 \\
\cos\left[\frac{\pi}{2} \nu \left(\frac{3}{2\pi} |\omega| - 1\right)\right] & \text{if } 2\pi/3 \leq |\omega| \leq 4\pi/3 \\
0 & \text{otherwise}
\end{cases}
\]

[error in Daubechies : $3/4\pi$ instead of $3/2\pi$ inside $\nu$]
where \( \nu \) is any infinitely differentiable non-negative function satisfying

\[
\nu(x) =
\begin{cases}
0 & \text{if } x \leq 0 \\
1 & \text{if } x \geq 1
\end{cases}
\]

smooth transition in \( \nu \) from 0 to 1 as \( x \) goes from 0 to 1

and

\[
\nu(x) + \nu(1 - x) = 1.
\]
fig 34: \( \nu(x) \) and \( \nu(1 - x) \)
fig 35: Fourier transform $\hat{\phi}(\omega)$ of the Meyer scaling function
Need to verify necessary properties for a scaling function:

(i)

\[ \sum_{k} |\hat{\phi}(\omega + 2\pi k)|^2 = \frac{1}{2\pi} \quad (21) \]

To see this, consider the two possible ranges of values of \( \omega \):
(a) \(|\omega + 2\pi k_1| \leq 2\pi/3\) for some \(k_1\). In that case (see diagram above):

\[
\hat{\phi}(\omega + 2\pi k_1) = \frac{1}{\sqrt{2\pi}}; \quad \hat{\phi}(\omega + 2\pi k) = 0 \text{ if } k \neq k_1
\]

since if \(|\omega + 2\pi k_1| \leq 2\pi/3\), then \(|\omega + 2\pi k| \geq 4\pi/3\) for \(k \neq k_1\). Thus (21) holds because there is only one non-zero term in that sum.
(b) $2\pi/3 \leq \omega + 2\pi k_1 \leq 4\pi/3$ for some $k_1$. In this case we also have

$$-4\pi/3 \leq \omega + 2\pi(k_1 - 1) \leq -2\pi/3.$$ 

Also, for all values $k \neq k_1$ or $k_1 - 1$, can calculate that

$$2\pi k \notin [-4\pi/3, 4\pi/3],$$

so

$$\hat{\phi}(\omega + 2\pi k) = 0.$$ 

So sum has only two non-zero terms:
$$2\pi \sum_k |\tilde{\phi}(\omega + 2\pi k)|^2 = 2\pi \left(|\tilde{\phi}(\omega + 2\pi k_1)|^2 + |\tilde{\phi}(\omega + 2\pi (k_1 - 1)|^2\right).$$

$$= \cos^2 \left[\frac{\pi}{2} \nu \left(\frac{3}{2\pi} |\omega + 2\pi k_1| - 1\right)\right] + \cos^2 \left[\frac{\pi}{2} \nu \left(\frac{3}{2\pi} |\omega + 2\pi (k_1 - 1)| - 1\right)\right]$$

$$= \cos^2 \left[\frac{\pi}{2} \nu \left(\frac{3}{2\pi} (\omega + 3k_1) - 1\right)\right] + \cos^2 \left[\frac{\pi}{2} \nu \left(-\frac{3}{2\pi} (\omega - 3k_1) + 2\right)\right]$$

$$= \cos^2 \left[\frac{\pi}{2} \nu \left(\frac{3}{2\pi} (\omega + 3k_1) - 1\right)\right] + \cos^2 \left[\frac{\pi}{2} \nu \left(1 - \frac{3}{2\pi} (\omega - 3k_1) + 2\right)\right]$$

$$= \cos^2 \left[\frac{\pi}{2} \nu \left(\frac{3}{2\pi} (\omega + 3k_1) - 1\right)\right] + \cos^2 \left[\frac{\pi}{2} \nu \left(\frac{3}{2\pi} (\omega + 3k_1) - 1\right)\right]$$

$$= \cos^2 \left[\frac{\pi}{2} \nu \left(\frac{3}{2\pi} (\omega + 3k_1) - 1\right)\right] + \sin^2 \left[\frac{\pi}{2} \nu \left(\frac{3}{2\pi} (\omega + 3k_1) - 1\right)\right]$$

$$= 1$$
Note that above $|\omega + 2\pi(k_1 - 1)| = -(\omega + 2\pi(k_1 - 1))$, since quantity in parentheses always negative for our range of $\omega$. In next to last equality have used $\cos\left(\frac{\pi}{2} - x\right) = \sin x$.

Note since cases (a), (b) cover all possibilities for $\omega$ (since they cover a range of size $2\pi$ for $\omega + 2\pi k_1$), we are finished proving (21).
Also need to verify:

(ii) 

\[ \hat{\phi}(\omega) = m_0(\omega/2)\hat{\phi}(\omega/2) \]

for some \(2\pi\)-periodic \(m_0(\omega/2)\). Indeed, looking at pictures:
fig 36: $\hat{\phi}(\omega)$ and $\hat{\phi}(\omega/2)$ (----)
ratio of these two looks like:

\[ \frac{\hat{\phi}(\omega)}{\hat{\phi}(\omega/2)} = \sqrt{2\pi} \hat{\phi}(\omega) \] in the interval \([-2\pi, 2\pi]\).
Note since ratio \( \frac{\hat{\phi}(\omega)}{\hat{\phi}(\omega/2)} = \sqrt{2\pi} \hat{\phi}(\omega) \) in \([-2\pi, 2\pi]\), we can define

\[
m_0(\omega/2) = \frac{\hat{\phi}(\omega)}{\hat{\phi}(\omega/2)} = \sqrt{2\pi} \hat{\phi}(\omega)
\]

if \( \omega \in [-2\pi, 2\pi] \).

Definition ambiguous when numerator and denominator are 0.
Definition also ambiguous for \( \omega \notin [-2\pi, 2\pi] \) since numerator and denominator both 0. So define \( m_0(\omega/2) \) by periodic extension of above for all real \( \omega \).

How to do that? Just add all possible translates of the bump \( \hat{\phi}(\omega) \) to make it \( 4\pi \)-periodic:

\[
m_0(\omega/2) = \sqrt{2\pi} \sum_k \hat{\phi}(\omega + 4\pi k).
\]
Check:

\[ m_0(\omega/2)\hat{\phi}(\omega/2) = \sqrt{2\pi} \sum_k \hat{\phi}(\omega + 4\pi k)\hat{\phi}(\omega/2) \]

\[ = \sqrt{2\pi} \hat{\phi}(\omega)\hat{\phi}(\omega/2) \]

\[ = \hat{\phi}(\omega) \]

where we have used the fact that \( \hat{\phi}(\omega + 4\pi k) \) has no overlap with \( \hat{\phi}(\omega/2) \) if \( k \neq 0 \).

[So we expect a full MRA.]
2. Construction of the Meyer wavelet

Standard construction:

\[ \hat{\psi}(\omega) = e^{i\omega/2}m_0(\omega/2 + \pi)\hat{\phi}(\omega/2) \]

\[ = e^{i\omega/2} \sum_k \hat{\phi}(\omega + 2\pi(2k + 1))\hat{\phi}(\omega/2) \]

\[ = e^{i\omega/2} \left[ \hat{\phi}(\omega + 2\pi) + \hat{\phi}(\omega - 2\pi) \right]\hat{\phi}(\omega/2) \]
[supports of 2d and 3d factors do not overlap for other values of \( k \); note \( \tilde{\phi} = \hat{\phi} \) since \( \hat{\phi} \) is real]
fig 38: $\hat{\phi}(\omega + 2\pi) + \hat{\phi}(\omega - 2\pi)$ (dashed) and $\hat{\phi}(\omega/2)$
fig 39: $\left[\hat{\phi}(\omega + 2\pi) + \hat{\phi}(\omega - 2\pi)\right]\hat{\phi}(\omega/2)$
Thus have 2 distinct regions:
(a) For $2\pi/3 \leq \omega \leq 4\pi/3$ we see in diagram that
$$e^{-i\omega/2}\hat{\psi}(\omega) = \sqrt{2\pi} \left[ \hat{\phi}(\omega + 2\pi) + \hat{\phi}(\omega - 2\pi) \right] \hat{\phi}(\omega/2)$$

$$= \hat{\phi}(\omega - 2\pi)$$

$$= \frac{1}{\sqrt{2\pi}} \cos \left[ \frac{\pi}{2} \nu \left( \frac{3}{2\pi} |\omega - 2\pi| - 1 \right) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \cos \left[ \frac{\pi}{2} \nu \left( -\frac{3}{2\pi} (\omega - 2\pi) - 1 \right) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \cos \left[ \frac{\pi}{2} \nu \left( -\frac{3}{2\pi} \omega + 2 \right) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \cos \left[ \frac{\pi}{2} \left[ 1 - \nu \left( 1 - \left( -\frac{3}{2\pi} \omega + 2 \right) \right) \right] \right]$$

$$= \frac{1}{\sqrt{2\pi}} \cos \left[ \frac{\pi}{2} \left[ 1 - \nu \left( \frac{3}{2\pi} \omega - 1 \right) \right] \right]$$
\[
\frac{1}{\sqrt{2\pi}} \sin \left[ \frac{\pi}{2} \nu \left( \frac{3}{2\pi} \omega - 1 \right) \right]
\]

So by symmetry same is true in \(-2\pi/3 \leq \omega \leq -4\pi/3\), so replace \(\omega\) by \(|\omega|\) above to get:

\[
e^{-i\omega/2} \hat{\psi}(\omega) = \frac{1}{\sqrt{2\pi}} \sin \left[ \frac{\pi}{2} \nu \left( \frac{3}{2\pi} |\omega| - 1 \right) \right]
\]

for \(2\pi/3 \leq |\omega| \leq 4\pi/3\)
(b) For $4\pi/3 \leq \omega \leq 8\pi/3$, we see from diagram (note $2\pi/3 \leq \omega/2 \leq 4\pi/3$):

$$e^{-i\omega/2}\hat{\psi}(\omega) = \sqrt{2\pi} \left[ \hat{\phi}(\omega + 2\pi) + \hat{\phi}(\omega - 2\pi) \right] \hat{\phi}(\omega/2)$$

$$= \hat{\phi}(\omega/2)$$

$$= \frac{1}{\sqrt{2\pi}} \cos \left[ \frac{\pi}{2} \nu \left( \frac{3}{2\pi} \omega/2 - 1 \right) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \cos \left[ \frac{\pi}{2} \nu \left( \frac{3}{4\pi} \omega - 1 \right) \right]$$
Again by symmetry same is true in $-8\pi/3 \leq \omega \leq -4\pi/3$, so replace $\omega$ by $|\omega|:\]

\[e^{-i\omega/2}\hat{\psi}(\omega) = \frac{1}{\sqrt{2\pi}} \cos \left[ \frac{\pi}{2} \nu \left( \frac{3}{4\pi} |\omega| - 1 \right) \right]\]

for $4\pi/3 \leq |\omega| \leq 8\pi/3$
Thus:
\[
\hat{\psi}(\omega) = \begin{cases} 
  e^{i\omega/2} \sin \left[ \frac{\pi}{2} \nu \left( \frac{3}{2\pi} |\omega| - 1 \right) \right], & \text{if } 2\pi/3 \leq |\omega| \leq 4\pi/3 \\
  e^{i\omega/2} \cos \left[ \frac{\pi}{2} \nu \left( \frac{3}{4\pi} |\omega| - 1 \right) \right], & \text{if } 4\pi/3 \leq |\omega| \leq 8\pi/3 \\
  0 & \text{otherwise}
\end{cases}
\]
Fig. 40: The wavelet Fourier transform $|\hat{\psi}(\omega)|$
Fig. 41: The Meyer wavelet $\psi(x)$
3. Properties of the Meyer wavelet

Note: If \( \nu \) is chosen as above and has all derivatives 0 at 0 and 1, can check that \( \hat{\psi}(\omega) \) is:
infinitely differentiable (since it is a composition of infinitely differentiable functions), and one can check that all derivatives are 0 from both sides at the break. For example, the derivatives coming in from the left at $\omega = \frac{2\pi}{3}$ are:

$$\frac{d^n}{d\omega^n} \hat{\psi}(\omega^-) \bigg|_{\omega=\frac{2\pi}{3}} = 0$$
and similarly

\[ \frac{d^n}{d\omega^n} \hat{\psi}(\omega^+) \bigg|_{\omega=\frac{2\pi}{3}} = 0 \]

(proof in exercises).

- supported (non-zero) on a finite interval
Lemma:

(a) If a function $\psi(x)$ has $n$ derivatives which are integrable, then the Fourier transform satisfies

$$|\hat{\psi}(\omega)| \leq K(1 + |\omega|)^{-n}.$$  \hfill (23)

Conversely, if (23) holds, then $\psi(x)$ has at least $n - 2$ derivatives.

(b) Equivalently, if $\hat{\psi}(\omega)$ has $n$ integrable derivatives, then

$$|\psi(x)| \leq K(1 + |x|)^{-n}$$  \hfill (24)
Conversely, if (24) holds, then $\widehat{\psi}(\omega)$ has at least $n - 2$ derivatives.

**Proof:** in exercises.

**Thus:** $\psi(x)$

- Decays at $\infty$ faster than any inverse power of $x$
- Is infinitely differentiable
Claim:

$$\psi_{jk}(x) = 2^{j/2} \psi(2^j x - k)$$

form an orthonormal basis for $L^2(\mathbb{R})$.

- Check (only to verify above results - we already know this to be true from our theory):

$$\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = \int_{-\infty}^{\infty} |\hat{\psi}(\omega)|^2 \, d\omega = 1$$
Pf:

\[
\int_{-\infty}^{\infty} |\hat{\psi}(\omega)|^2 \ d\omega = \frac{1}{2\pi} \left( \int_{\frac{2\pi}{3} \leq |\omega| \leq \frac{4\pi}{3}} \ d\omega \sin^2 \left[ \frac{\pi}{2} \nu \left( \frac{3}{2\pi} |\omega| - 1 \right) \right] \right.
\]

\[
+ \int_{\frac{4\pi}{3} \leq |\omega| \leq \frac{8\pi}{3}} \ d\omega \cos^2 \left[ \frac{\pi}{2} \nu \left( \frac{3}{4\pi} |\omega| - 1 \right) \right] \right)
\]

[getting rid of the $| \cdot |$ and doubling; changing vars. in second integral]
\[
\frac{1}{\pi} \left( \int_{\frac{2\pi}{3} \leq \omega \leq \frac{4\pi}{3}} d\omega \sin^2 \left[ \frac{\pi}{2} \nu \left( \frac{3}{2\pi} \omega - 1 \right) \right] \right.
\]
\[
+ 2 \int_{\frac{2\pi}{3} \leq \omega \leq \frac{4\pi}{3}} d\omega \cos^2 \left[ \frac{\pi}{2} \nu \left( \frac{3}{2\pi} \omega - 1 \right) \right] \bigg) 
\]
\[
= \frac{1}{\pi} \left( \int_{\frac{2\pi}{3} \leq \omega \leq \frac{4\pi}{3}} d\omega \left\{ \sin^2 \left[ \frac{\pi}{2} \nu \left( \frac{3}{2\pi} \omega - 1 \right) \right] + 2 \cos^2 \left[ \frac{\pi}{2} \nu \left( \frac{3}{2\pi} \omega - 1 \right) \right] \right\} \right)
\]
\[
= \frac{1}{\pi} \left( \int_{\frac{2\pi}{3} \leq \omega \leq \frac{4\pi}{3}} d\omega \left\{ 1 + \cos^2 \left[ \frac{\pi}{2} \nu \left( \frac{3}{2\pi} \omega - 1 \right) \right] \right\} \right)
\]
\[
= 552
\]
[letting $s = \frac{3}{2\pi} \omega - 1 \Rightarrow \omega = \frac{2\pi}{3(s + 1)}$]

$$= \frac{2}{3} \left( \int_0^1 ds \left( 1 + \cos^2 \left[ \frac{\pi}{2} \nu(s) \right] \right) \right)$$

$$= \frac{2}{3} \left( \int_0^{1/2} ds \left( 1 + \cos^2 \left[ \frac{\pi}{2} \nu(s) \right] \right) + \int_{1/2}^1 ds \left( 1 + \cos^2 \left[ \frac{\pi}{2} \nu(s) \right] \right) \right)$$

$$= \frac{2}{3} \left( \int_0^{1/2} ds \left( 1 + \cos^2 \left[ \frac{\pi}{2} \nu(s) \right] \right) + \int_0^{1/2} ds \left( 1 + \cos^2 \left[ \frac{\pi}{2} \nu(s + 1/2) \right] \right) \right)$$

553
[using \( \nu(s + 1/2) = 1 - \nu(1/2 - s) \)]

\[
= \frac{2}{3} \left( \int_0^{1/2} ds \left( 1 + \cos^2 \left( \frac{\pi}{2} \nu(s) \right) \right) + \int_0^{1/2} ds \left( 1 + \cos^2 \left( \frac{\pi}{2} \nu(1/2 - s) \right) \right) \right)
\]

\[
= \frac{2}{3} \left( \int_0^{1/2} ds \left( 1 + \sin^2 \left( \frac{\pi}{2} \nu(1/2 - s) \right) \right) \right)
\]

\[
s \rightarrow 1/2 - s \quad \xrightarrow{s \rightarrow 1/2 - s} \quad \frac{2}{3} \left( \int_0^{1/2} ds \left( 1 + \cos^2 \left( \frac{\pi}{2} \nu(s) \right) \right) + \int_0^{1/2} ds \left( 1 + \sin^2 \left( \frac{\pi}{2} \nu(s) \right) \right) \right)
\]

\[
= \frac{2}{3} \left( \int_0^{1/2} ds (2 + 1) \right) = 1
\]

554
To show in another way that they form an orthonormal basis, sufficient to show that for arbitrary $f \in L^2(\mathbb{R})$,

$$\sum_{j,k} |\langle \psi_{jk}, f \rangle|^2 = \int_{-\infty}^{\infty} |f(x)|^2 \, dx$$
[this is a basic analytic theorem].

Now note:

\[
\sum_{j,k} \left| \langle \psi_{jk}, f \rangle \right|^2 = \sum_{j,k} \left| \int dx \, \overline{\psi_{jk}(x)} f(x) dx \right|^2
\]

\[
= \sum_{j,k} \left| \int d\omega \, \hat{f}(\omega) \overline{\hat{\psi}_{jk}(\omega)} \right|^2.
\]
Note if

\[ \psi_{jk}(x) = 2^{j/2} \psi(2^j x - k). \]

Then as usual:

\[ \hat{\psi}_{jk}(\omega) = 2^{-j/2} \hat{\psi}(2^{-j} \omega) e^{-i2^{-j}k\omega}. \]

Plug this in above and can do calculation to show (we won't do the calculation):

\[ \sum_{j,k} |\langle f, \psi_{jk} \rangle|^2 = \int_{-\infty}^{\infty} dx |f(x)|^2, \]

as desired.
CONCLUSION:

The wavelets

$$\psi_{jk}(x) \equiv 2^{j/2} \psi(2^j x - k)$$

form an orthonormal basis for the square integrable functions on the real line.
4. Daubechies wavelets:

Recall that one way we have defined wavelets is by starting with the scaling (pixel) function $\hat{\phi}(x)$. Recall it satisfies:

$$\hat{\phi}(\omega) = m_0(\omega/2)\hat{\phi}(\omega/2)$$

for all $\omega$, where $m_0(\omega)$ is some periodic function. If we use $m_0$ as the starting point, recall we can write

$$\hat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}} \prod_{j=1}^{\infty} m_0(\omega/2^j).$$  \hspace{1cm} (25)
Recall $m_0$ is periodic, and so has Fourier series:

$$m_0(\omega) = \sum_k a_k e^{-ik\omega}.$$ 

If $m_0$ satisfies $|m_0(\omega)|^2 + |m_0(\omega + \pi)|^2 = 1$, then it is a candidate for construction of wavelets and scaling functions.

For Haar wavelets, recall $m_0(\omega) = e^{i\omega/2}\cos \omega/2$, so we could plug into (25) to get $\hat{\phi}$, and then use previous formulas to get wavelet $\psi(x)$. 

560
If we start with a function $m_0(\omega)$, when does (25) lead to a genuine wavelet? Check conditions:

(1)

$$\hat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}} \prod_{j=1}^{\infty} m_0(\omega/2^j)$$

$$= \frac{1}{\sqrt{2\pi}} m_0(\omega/2) \prod_{j=2}^{\infty} m_0(\omega/2^j)$$
\[ = m_0(\omega/2) \frac{1}{\sqrt{2\pi}} \prod_{j=1}^{\infty} m_0(\omega/2^{j+1}) \]

\[ = m_0(\omega/2) \hat{\phi}(\omega/2) \quad \text{(26)} \]

Recall this implies that \( V_j \subset V_{j+1} \) where

\[ V_j = \left\{ \sum_{k=-\infty}^{\infty} a_k \phi_{jk}(x) \left| \sum_k |a_k|^2 < \infty \right. \right\} \]

(usual definition) with \( \phi_{jk}(x) = 2^{j/2} \phi(2^j x - k) \)
The second condition we need to check is that translates of $\phi$ orthonormal, i.e.,

$$\sum_k |\hat{\phi}(\omega + 2\pi k)|^2 = \frac{1}{2\pi}.$$ 

If

$$m_0(\omega) = \text{finite Fourier series}$$

$$= \sum_{k=-N}^{N} a_k e^{-i\omega k} = \text{trigonometric polynomial}$$
There is a simple condition which guarantees condition (2) holds.

Theorem (Cohen, 1990): If the trigonometric polynomial $m_0$ satisfies $m_0(0) = 1$ and

$$|m_0(\omega)|^2 + |m_0(\omega + \pi)|^2 = 1$$

(our standard condition on $m_0$), and also $m_0(\omega) \neq 0$ for $|\omega| \leq \pi/3$, then condition (2) above is satisfied by

$$\hat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}} \prod_{j=1}^{\infty} m_0(\omega/2^j)$$
Proof: Daubechies, Chapter 6.

Since condition (1) is also automatically satisfied, this means $\phi$ is a scaling function which will lead to a full orthonormal basis using our algorithm for constructing wavelets.

Another choice of $m_0$ is:

$$m_0(\omega) = \frac{1}{8}[(1 + \sqrt{3}) + (3 + \sqrt{3})e^{-i\omega} + (3 - \sqrt{3})e^{-2i\omega} + (1 - \sqrt{3})e^{-3i\omega}]$$

(Fourier series with finite number of terms).
Fig 42: Real (symmetric) and imaginary (antisymmetric) parts of $m_0(\omega)$
To check Cohen's theorem satisfied:

(i) Equation (27) satisfied (see exercises).

(ii) If \( m_0(\omega) = \text{Re} \ m_0(\omega) + i \ \text{Im} \ m_0(\omega) \),

\[
|m_0(\omega)|^2 = |\text{Re} \ m_0(\omega)|^2 + |\text{Im} \ m_0(\omega)|^2 \neq 0
\]

for \(|\omega| \leq \pi/3\), as can be seen from graph above.

So: conditions of Cohen's theorem are satisfied.

In this case if we define scaling function \( \phi \) by computing infinite product (25) (perhaps numerically), and then
use our standard procedure to construct wavelet \( \psi(x) \), we get:
fig 43: pictures of $\phi$ and $\psi$
Note meaning of $m_0$: In terms of the original wavelet, this states

$$\phi(x) = \frac{1}{4} \left[(1 + \sqrt{3})\phi(2x) + (3 + \sqrt{3})\phi(2x - 1) + (3 - \sqrt{3})\phi(2x - 2) + (1 - \sqrt{3})\phi(2x - 3)\right]$$

(see (26) above). Note this equation gives the information we need on $\phi$, since it determines $m_0(\omega)$. 

570