MA 412 Complex Analysis

Exercises for Chapter V: Taylor and Laurent Series

Ex. 1: Prove the next theorem

Suppose that \( z_n = x_n + i y_n \), for \( n = 1, 2, \ldots \) and \( S = X + i Y \). Then \( \Sigma_{n=1}^{\infty} z_n = S \), if and only if \( \Sigma_{n=1}^{\infty} x_n = X \) and \( \Sigma_{n=1}^{\infty} y_n = Y \).

Ex. 2: Show in two ways (definition of limit and Thm. V.2.) that the sequence 
\( z_n = -2 + i(-1)^n/n \) for \( n = 1, 2, \ldots \), converges to \(-2\).

Ex. 3: By considering the remainders \( \rho_N(z) \), verify that

\[
\Sigma_{n=1}^{\infty} z^n = \frac{z}{1 - z}
\]

when \( \|z\| < 1 \).

Hint: Use identity

\[
1 + z + z^2 + \ldots + z^N = \frac{1 - z^{N+1}}{1 - z}
\]

for \( z \neq 1 \), to show that \( \rho_N(z) = z^{N+1}/(1 - z) \). Then use condition (*).

Ex. 4: Write \( z = re^{i\theta} \), where \( 0 < r < 1 \) in the summation formula

\[
\Sigma_{n=1}^{\infty} z^n = \frac{z}{1 - z}.
\]

Then using Thm. V.4., show that

\[
\Sigma_{n=1}^{\infty} r^n \cos(n\theta) = \frac{r \cos \theta - r^2}{1 - 2r \cos \theta + r^2}
\]

and,

\[
\Sigma_{n=1}^{\infty} r^n \sin(n\theta) = \frac{r \sin \theta}{1 - 2r \cos \theta + r^2},
\]

when \( 0 < r < 1 \).

Section 7.1: 2,3.

Ex. 5: Show

1. \[
\frac{1}{1 + z} = \Sigma_{n=0}^{\infty} (-1)^n z^n, \quad \|z\| < 1,
\]

2. \[
\frac{1}{z} = \Sigma_{n=0}^{\infty} (-1)^n (z - 1)^n, \quad \|z - 1\| < 1.
\]
Section 7.2: 1,2(d),3,5,7,8,15.

Ex. 6: Expand cos \( z \) in Taylor series about \( \pi/2 \).

Answer:
\[
\sum_{n=0}^{\infty} (-1)^{n-1} \frac{(z - \pi/2)^{2n+1}}{(2n+1)!}
\]

Ex. 7: Use the formula \( \sin z = (e^{iz} - e^{-iz})/(2i) \) to find the Taylor series for \( \sin z \).

Ex. 8: Find the Taylor series of
1. \( e^z/z^2 \),
2. \( \sin(z^2)/z^4 \).

Ex. 9: Find the Laurent series of
\[
f(z) = z^2 \sin\left(\frac{1}{z}\right).
\]

Ex. 10: Derive the Laurent series for
\[
\frac{e^z}{(z+1)^2}, \quad 0 < \|z + 1\| < \infty.
\]

Ex. 11: Represent the function \( f(z) = (z + 1)/(z - 1) \) by
1. a Taylor series centered about 0, and
2. a Laurent series for the domain \( 1 < \|z\| < \infty \).

Section 7.3: 1,3,4,5,8,9,11,12.