MA 412 Complex Analysis

The Complete Review

The next list presents the most important results for the final exam. The list is not exhaustive, but it should give you a good idea of what to expect in the exam. You should know the statements of the theorems and know how to apply them, but you don’t need to know the proofs (unless indicated by *). The checklist is divided in three parts, all of them equally important.

Part I: The Complex Plane & Analytic Functions

1. Definitions and properties of the absolute value and conjugation.
2. The triangle inequality
   \[|z_1 + z_2| \leq |z_1| + |z_2|,\]
   \[|z_1 - z_2| \geq ||z_1| - |z_2||.\]
3. Polar coordinates: you should be able to convert rectangular coordinates to polar, and vice versa. Also Euler’s and De Moivre’s formulas:
   \[e^{i\theta} = \cos \theta + i \sin \theta,\]
   \[e^{in\theta} = \cos(n\theta) + i \sin(n\theta).\]
4. How to find the roots of a complex number and be able to draw a picture of where these roots lie in the complex plane.
5. Definition of limit in terms of \(\epsilon\) and \(\delta\). How to compute limits and justify when the limit does not exist.
6. Definition of continuity in the complex case.
7. Definition of complex derivative and be able to verify if the derivative exists in a neighborhood of a point.
9. Definition of harmonic functions and harmonic conjugate. How to compute the harmonic conjugate for a given function \(u(x, y)\).
10. Theorems you should know
    (a) **Theorem** Suppose that
        \[f(z) = u(x, y) + iv(x, y)\]
and that \( f'(z) \) exists at a point \( z_0 = x_0 + iy_0 \). Then the first-order partial derivatives of \( u \) and \( v \) exist at \( (x_0, y_0) \) and they must satisfy the Cauchy-Riemann equations

\[
    u_x = v_y \quad \text{and} \quad v_x = -u_x
\]

there. Also the derivative of \( f \) can be written as

\[
    f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).
\]

(b) **Theorem** Let the function

\[
    f(z) = u(x, y) + iv(x, y)
\]

be defined throughout an \( \epsilon - \)neighborhood of a point \( z_0 = x_0 + iy_0 \). Suppose that the first-order partial derivatives of the functions \( u \) and \( v \) exist and are continuous at the point \( (x_0, y_0) \). Then \( f'(z_0) \) exists.

(c) **Theorem** If a function \( f(z) = u(x, y) + iv(x, y) \) is analytic in a domain \( D \), then the function \( u \) and \( v \) are harmonic in \( D \).

### Part II: Elementary Functions & Complex Integration

1. Definitions and properties of complex exponential, complex logarithm and trigonometric functions. Branches of the logarithmic function. Solving equations that involve the exponential, the logarithmic and complex powers.

2. Definition of contours and types of contours.

3. Integrals of complex functions of a real variable

\[
    \int_C f(z)dz = \int_a^b f(z(t))z'(t)dt
\]

and the inequality

\[
    \left| \int_a^b f(t) \, dt \right| \leq \int_a^b |f(t)| \, dt.
\]

How to compute the length of a contour.

4. Contour integrals for complex functions of a complex variable, properties and the ML inequality:

\[
    \left| \int_C f(z) \, dz \right| \leq ML,
\]

where \( M \) is the maximum value of \( |f| \) on the contour \( C \), and \( L \) is the length of the contour.

5. Theorems you **should** know
(a) **Theorem** Suppose that a function $f$ is continuous on a domain $D$. Then if any of the following statements are true, then so the others:

i. $f$ has an analytic antiderivative $F$ in $D$

ii. the integrals of $f(z)$ along contours lying entirely in $D$ and extending from a fixed point $z_1$ to any other fixed point $z_2$ all have the same value.

iii. the integrals of $f(z)$ around closed contours lying entirely in $D$ all have value zero.

(b) **Theorem** (Cauchy-Goursat) If a function $f$ is analytic at all points interior and on a simple closed contour $C$, then

$$\int_C f(z) \, dz = 0.$$ 

(c) **Theorem** (Deformation Principle) Let $C_1$ and $C_2$ be positively oriented simple closed contours, where $C_2$ is interior to $C_1$. If a function $f$ is analytic in the region between the two contours, then

$$\int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz.$$  

(you should also be able to generalize this theorem to finitely many contour lying inside $C_1$ (general Cauchy-Goursat Theorem)).

(d) **Theorem** (Cauchy Integral Formula) Let $f$ be analytic everywhere within and on a simple closed contour $C$, taken in the positive sense. If $z_0$ is any point interior to $C$, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} \, dz,$$

and furthermore

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz.$$ 

(e) **Theorem** If $f$ is analytic in a neighborhood of $z_0$, then all its derivatives are analytic functions in the same neighborhood of $z_0$.

(f) **Theorem** * (Morera) If $f$ is a continuous function over a domain $D$ and such that

$$\int_C f(z) \, dz = 0$$  

for all simple closed contour $C$ that lies completely inside $D$, then $f$ is analytic in $D$.

(g) **Theorem** * (Liouville) If $f$ is both entire and bounded, then $f$ is a constant function.
(h) **Theorem** *(Fundamental Theorem of Algebra)* Any polynomial
\[ p(z) = a_0 + a_1 z + \ldots + a_n z^n, \quad (a_n \neq 0) \]
of degree \( n \) \( (n > 0) \) has at least one root, that is, there exists a point \( z_0 \) such that \( p(z_0) = 0 \).

**Part III: Taylor, Laurent Series & Residue Theory**

1. Definition of the Taylor and Maclaurin power series for an analytic function \( f \) in a neighborhood of a point \( z_0 \).

2. You should know the formulas for the coefficients in the Laurent series
\[ a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz \]
for \( n = 0, 1, \ldots \), and
\[ b_n = \frac{1}{2\pi i} \int_C \frac{(z - z_0)^{n-1} f(z)}{z - z_0} \, dz \]
for \( n = 1, 2, \ldots \).

3. How to multiply and compute derivatives of a power series.

4. Definition of the three types of singularities (removable, pole and essential) and how to distinguish them in terms of the power expansion of the function around the singular point.

5. Definition of residue of a function at a point and how to find it.

6. How to find the order of a zero and the order of a pole.

7. Theorems you **should** know

   (a) **Theorem** *(Cauchy’s Residue Theorem)* Let \( C \) be a positively oriented simple closed contour. If a function is analytic inside and on \( C \) except for a finite number of singular points \( z_k \) \( (k = 1, 2, \ldots, n) \) inside \( C \), then
   \[ \int_C f(z) \, dz = 2\pi i \sum_{k=0}^{n} \text{Res}_{z=z_k} f(z). \]

   (b) **Theorem** If a function \( f \) has a pole of order \( m \leq 1 \) at \( z_0 \), then the residue of \( f \) at \( z_0 \) is
   \[ \frac{1}{(m-1)!} \lim_{z \to z_0} \left( \frac{d^{(m-1)}}{dz^{(m-1)}} (z - z_0)^m f(z) \right). \]
(c) If a function
\[ f(z) = \frac{p(z)}{q(z)} \]
has a pole of order 1 at \( z_0 \), then we can compute
\[ \text{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)}. \]

(d) **Theorem** If \( p \) and \( q \) are polynomials that satisfy
\begin{enumerate}
  \item \( q(x) \neq 0 \) for all real numbers \( x \),
  \item \( \deg q(x) \geq \deg p(x) + 2 \),
\end{enumerate}
then
\[ \int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \, dx = 2\pi i \sum \text{Res}_{z=z_k} \frac{p(z)}{q(z)} \]
where the sum is taken only for those singularities \( z_k \) with positive imaginary part.

For further study:
- **Complex Variables and Applications**, Brown and Churchill.
- **Complex Analysis**, Serge Lang.
- **Complex Analysis**, Ahlfors.

"We don't learn mathematics, we just get used to them"
— John Von Neumann.