MA 412 Complex Analysis
Solutions to Quiz # 2

1. Consider the following set:

\[ \{re^{i\theta} | 0 < r < 1 \text{ and } -\pi/2 < \theta < \pi/2 \}. \]

Sketch this set and state if is open or closed, connected or disconnected, a domain or a region, bounded or unbounded.

**Answer** The region given is open and connected, hence is a domain. It is also a region by definition, and is clearly bounded.

2. Express \( f(z) = z^5 + z^3 \) in polar coordinates of the form \( u(r, \theta) + iv(r, \theta) \).

**Answer** Let \( z = re^{i\theta} \). As \( \overline{z} = re^{-i\theta} \), we have

\[ f(z) = z^5 + \overline{z}^3 = r^5 e^{5i\theta} + r^3 e^{-3i\theta}. \]

By applying De Moivre’s formula,

\[ f(z) = r^5 (\cos 5\theta + i \sin 5\theta) + r^3 (\cos 3\theta - i \sin 3\theta). \]

After equating real and imaginary parts, we obtain

\[ f(z) = (r^5 \cos 5\theta + r^3 \sin 3\theta) + i(r^5 \sin 5\theta + r^3 \sin 3\theta). \]

3. Let \( f(z) = \overline{z} \), \( g(z) \) be a nonconstant continuous function in \( \mathbb{C} \), and \( F(z) = g(z) \).
1. Show that $f$ is continuous in $\mathbb{C}$.
2. Show that $F$ is continuous in $\mathbb{C}$.
3. Show that $f$ is nowhere differentiable.
4. Show that $F$ is not analytic in $\mathbb{C}$.

**Answers**

1. A function $f(z) = u(x, y) + iv(x, y)$ is continuous at $z_0 = x_0 + iy_0$ if and only if $u(x, y)$ and $v(x, y)$ are continuous at $(x_0, y_0)$. For the given function, $u(x, y) = x$ and $v(x, y) = -y$ are clearly continuous at any $(x, y)$. It follows that $f(z) = z$ is continuous at any $z \in \mathbb{C}$.

2. By hypothesis, $g(z)$ is continuous in all $\mathbb{C}$, so is $f(z)$. Since the composition of two continuous functions is continuous, it follows that $F(z) = f \circ g(z)$ is continuous in all $\mathbb{C}$.

3. Using the contrapositive statement of the Cauchy-Riemann theorem for analytic functions, we just need to show that the functions $u(x, y) = x$ and $v(x, y) = y$ do not hold the Cauchy-Riemann equations at any point. Indeed,

$$u_x(x, y) = 1 \text{ and } v_y(x, y) = -1,$$

at any $(x, y)$, so $u_x \neq v_y$ and we conclude that $f$ is nowhere differentiable.

4. Assume $g(z) = a(x, y) + ib(x, y)$ is a non-constant, continuous and analytic function in $\mathbb{C}$. Hence the CR equations hold for $a$ and $b$, that is

$$a_x = b_y \text{ and } b_x = -a_x. \quad (1)$$

Write $F(z) = U(x, y) + iV(x, y)$. Then $U = a$ and $V = -b$ at any point $(x, y)$. If $F$ happens to be analytic, then, the CR equations should also hold. That is $U_x = V_y$ and $V_x = -U_y$. In terms of $a$ and $b$, we get

$$a_x = -b_y \text{ and } -b_x = -a_y \quad (2)$$

Combining expressions (1) and (2), we obtain

$$a_x = -b_y = b_y \text{ if and only if } b_y = 0, \text{ and so } a_x = 0$$

$$a_y = b_x = -b_x \text{ if and only if } b_x = 0, \text{ and so } a_y = 0$$

That is, the imaginary and the real parts of $g$ are constant, which is a contradiction. Hence, $F$ is not analytic in $\mathbb{C}$.

4. Let $f(z) = e^z \cos y + i e^z \sin y$. Show that $f$ and $f'$ are analytic in $\mathbb{C}$.
0.5 pts.  
**Answer** Let \( u(x, y) = e^x \cos y \) and \( v(x, y) = e^x \sin y \). \( f \) is analytic if \( f'(z) \) exists in a neighborhood of \( z \), but \( f' \) exists if \( u \) and \( v \) have continuous first-order partial derivatives and hold the CR equations. As \( e^x, \cos y \) and \( \sin y \) are continuous in \( \mathbb{R}^2 \) their product are also continuous in \( \mathbb{R}^2 \). Hence \( u_x, u_y, v_x \) and \( v_y \) are continuous in the real plane. Moreover

\[
 u_x = e^x \cos y = v_y \quad \text{and} \quad u_y = -e^x \sin y = -v_x
\]

so \( f' \) exists at any \( z = x + iy \). Hence \( f \) is analytic in all \( \mathbb{C} \). As \( f'(z) = u(x, y) + iv(x, y) \) for all \( (x, y) \) then \( f' = f \) and it follows that \( f' \) is also analytic.

5. Suppose \( v \) is harmonic conjugate of \( u \) and \( u \) is harmonic conjugate to \( v \). Show that \( u \) and \( v \) must be constant functions.

1 pts.  
**Answer** By definition, \( v \) is harmonic conjugated to \( u \) if \( \Delta u = \Delta v = 0 \) and \( u_x = v_y, u_y = -v_x \) hold. On the other hand, as \( u \) is harmonic conjugated to \( v \), we also have \( v_x = u_y \) and \( v_y = -u_x \). Then,

\[
 u_y = -v_x = v_x \quad \text{implies} \quad v_x = 0 \quad \text{and} \quad u_y = 0
\]

and

\[
 u_x = v_y = -v_y \quad \text{implies} \quad v_y = 0 \quad \text{and} \quad u_x = 0.
\]

Hence, \( u(x, y) \) and \( v(x, y) \) are constant functions.

6. Let \( u(x, y) = 2x(1 - y) \). Show that \( u \) is harmonic and find a harmonic conjugated.

1.5 pts.  
**Answer** Let \( u(x, y) = 2(1 - y) \). As \( u_{xx} = u_{yy} = 0 \) for all \( (x, y) \), we have \( \Delta u = 0 \), so \( u \) is harmonic in all the real plane. If \( v \) is harmonic conjugate of \( u \), then by the CR equations, \( v_y(x, y) = 2(1 - y) \). Integrating in terms of \( y \) we obtain

\[
 v(x, y) = \int (2(1 - y))dy = 2y - y^2 + \phi(x).
\]

Also \( u_y = -2x = -v_x \), that is \( v_x = 2x = \phi'(x) \). Integrating in terms of \( x \), we obtain that \( \phi(x) = x^2 + c \), for \( c \in \mathbb{R} \). Thus,

\[
 v(x, y) = 2y - y^2 + x^2 + c.
\]

We can easily check that \( \Delta v(x, y) = 0 \).