Problem 1
Probability densities integrate to 1. The trick here is to rewrite these in terms of known probability densities.

a) Write it as a normal density with mean \( \mu = 0 \) and standard deviation \( \sigma = \frac{1}{2} \).

\[
\int_{-\infty}^{\infty} e^{-2x^2} \, dx = \sqrt{2\pi} \left( \frac{1}{2} \right) \left[ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \left( \frac{1}{2} \right)} e^{-\frac{x^2}{2}} \, dx \right] = \sqrt{\frac{\pi}{2}} \cdot 1 = \sqrt{\frac{\pi}{2}}
\]

b) Write it as a Gamma(\( \alpha, \lambda \)) distribution with \( \alpha = \frac{1}{2} \) and \( \lambda = 1 \).

\[
\int_{0}^{\infty} x^{-1/2}e^{-x} \, dx = \frac{\Gamma(1/2)}{1^{1/2}} \left[ \int_{0}^{\infty} x^{1/2-1}e^{-x} \, dx \right] = \Gamma(1/2) \cdot 1 = \sqrt{\pi}
\]

You may have also noticed that this integral IS the gamma function of \( 1/2 \).

Problem 2
Since \( X \) and \( Y \) are Poisson, \( E(X) = \lambda = 1 \), \( \text{Var}(X) = \lambda_i = 1 \), \( E(Y) = \lambda_2 \) and \( \text{Var}(Y) = \lambda_2 \). Now,

\[
E(X + Y) = E(X) + E(Y) = \lambda_1 + \lambda_2 = 3 \\
\text{Var}(2X + Y) = 2^2 \text{Var}(X) + \text{Var}(Y) = 4\lambda_1 + \lambda_2 = 6 \\
\]

Since \( X, Y \) are independent

Subtracting these two equations gives \( 3\lambda_1 = 3 \Rightarrow \lambda_1 = 1 \) and \( \lambda_2 = 3 - \lambda_1 = 2. \)

Problem 3
a) We’ll start by finding \( E(X) \) and \( E(Y) \) and \( E(XY) \).

\[
E(X) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xf_{X,Y}(x,y) \, dx \, dy = \int_{0}^{1} \int_{0}^{1} x \left( \frac{3}{2} \right) (x^2 + y^2) \, dx \, dy = \left( \frac{3}{2} \right) \int_{0}^{1} \left( \frac{1}{4} + \frac{y^2}{2} \right) \, dy = \frac{5}{8}
\]

\[
E(Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} yf_{X,Y}(x,y) \, dx \, dy = \int_{0}^{1} \int_{0}^{1} y \left( \frac{3}{2} \right) (x^2 + y^2) \, dy \, dx = \left( \frac{3}{2} \right) \int_{0}^{1} \left( \frac{1}{4} + \frac{x^2}{2} \right) \, dx = \frac{5}{8}
\]

\[
E(X Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x y f_{X,Y}(x,y) \, dx \, dy = \int_{0}^{1} \int_{0}^{1} x y \left( \frac{3}{2} \right) (x^2 + y^2) \, dx \, dy = \left( \frac{3}{2} \right) \int_{0}^{1} \left( \frac{1}{4} + \frac{y^3}{2} \right) \, dy = \frac{3}{8}
\]

So, using one of the formulas for covariance,

\[
\text{Cov}(X, Y) = E(XY) - (E(X))(E(Y)) = \frac{3}{8} - \left( \frac{5}{8} \right)^2 = -\frac{1}{64} = -0.0156
\]

b) We start by finding the marginal density \( f_Y(y) \)

\[
f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx = \int_{0}^{1} \frac{3}{2} (x^2 + y^2) \, dx = \frac{1}{2} + \frac{3y^2}{2}, \quad 0 < y < 1
\]
Then, the conditional density $f_{X|Y}(x|y) = f_{X,Y}(x,y)/f_Y(y)$, and the conditional density of $X$ given $Y = 1/2$ is

$$f_{X|Y}(x|1/2) = \frac{f_{X,Y}(x,1/2)}{f_Y(1/2)} = \frac{\left(\frac{3}{2}\right)^2}{\frac{1}{2} + \frac{x(1/2)^2}{2}} = \frac{3}{7} + \frac{12}{7}x^2, \quad 0 < x < 1$$

So,

$$E(X|Y = 1/2) = \int_{-\infty}^{\infty} x f_{X|Y}(x|1/2) dx = \int_0^1 \left(\frac{3x}{7} + \frac{12x^3}{7}\right) dx = \frac{9}{14} = 0.643$$

**Problem 4**

a) If $N$ people play, the chance of winning is $1/N$. So given $N$, $X$ has the following distribution

$$X = \begin{cases} \frac{N}{2} & \text{with probability } \frac{1}{N} \\ 0 & \text{with probability } 1 - \frac{1}{N} \end{cases} \quad (1)$$

So,

$$E(X|N) = \left(\frac{N}{2}\right) \cdot \frac{1}{N} + 0 \cdot \left(1 - \frac{1}{N}\right) = \frac{1}{2} \cdot \frac{1}{N}$$

and from the law of total expectation,

$$EX = E[E(X|N)] = E\left(\frac{1}{2}\right) = \frac{1}{2} = 50 \text{ cents}$$

b) From the law of total variance,

$$\text{Var}X = E\text{Var}(X|N) + \text{Var}E(X|N) \quad (3)$$

From (1) and (2)

$$\text{Var}(X|N) = E(X^2|N) - (E(X|N))^2 = \left[\frac{N^2}{4} \cdot \frac{1}{N} + 0 \cdot \left(1 - \frac{1}{N}\right)\right] - \left(\frac{1}{2}\right)^2 = \frac{N-1}{4}$$

So, (3) becomes

$$\text{Var}X = E\left(\frac{N-1}{4}\right) + \text{Var}\left(\frac{1}{2}\right) = EN - \frac{1}{4} + 0 = \frac{499}{4} = 124.75 \quad (4)$$

(You ended up not even using the standard deviation of $N$).