Problem 1 The sample space $\Omega$ is all groups of 3 people from the office. Every group of 3 people in the office has an equally likely chance of getting picked, so the size of our sample space is $|\Omega| = \binom{16}{3}$. For both a) and b), the solution is of the form

$$\frac{\text{Number of favorable outcomes}}{|\Omega|}$$

a) Let $A$ be the event that Jim and Pam are chosen. The 3rd person can be any other member of the office, so

$$P(A) = \frac{\binom{2}{1} \cdot \binom{16-2}{1}}{\binom{16}{3}} = \frac{\binom{14}{1}}{\binom{16}{3}} = \frac{14 \cdot 3 \cdot 2}{16 \cdot 15 \cdot 14} = 0.025$$

b) Let $B$ be the event that Dwight is chosen and Jim is not. So, we want to pick Dwight, not pick Jim, and pick 2 from among the 14 others:

$$P(B) = \frac{\binom{1}{1} \cdot \binom{14}{2}}{\binom{16}{3}} = \frac{14 \cdot 13 \cdot 3}{16 \cdot 15 \cdot 14} = 0.1625$$

Problem 2

In a group of $N$ people, the number with a tattoo follows a Binomial distribution with parameters $N$ and $p = 0.24$.

a) $P[\text{exactly 2}] = \binom{10}{2} p^2 (1-p)^{10-2} = 0.2885$

b) In a group of $N$ people, the probability that at least one person has a tattoo is $1 - P[\text{Nobody has a tattoo}]= 1 - (1-p)^N$. We want to find $N$ so this probability is $> 95\%$. So, we want

$$1 - (1-p)^N > 0.95 \Rightarrow (1-p)^N < 0.05 \Rightarrow N > \frac{\log(0.05)}{\log(1-0.24)} = 10.916$$

So we must take at least $N = 11$ people.

Problem 3

a) In $t = 365$ days, the distribution is Poisson$(365\lambda) = \text{Poisson}(3,650)$

b) The random variable $X \sim \text{Poisson}(3,650)$ can be thought of as a sum of 365 independent Poisson$(10)$ random variables, each with mean 10 and variance 10 (since they have a Poisson distribution). This represents the sum of the number of people coming in each day, this is an independent sum since the Poisson process has independent increments. Thus, the central limit implies

$$P[X > 3750] \approx P \left[ Z > \frac{3750 - 365 \cdot 10}{\sqrt{365 \cdot \sqrt{10}}} \right] = P[Z > 1.66] = 1 - \Phi(1.66) = 1 - 0.9515 = 0.0485.$$
Problem 4

Method 1: PDF Method We have \( f_U(u) = 1 \) if \( 0 < u < 1 \) and 0 otherwise. Also,

\[
X = \ln \left( \frac{1}{U} \right) \Rightarrow U = e^{-X}
\]

Since \( U \in (0, 1) \), \( 1/U \in (1, \infty) \) and \( \ln(1/U) \in (0, \infty) \), so the range of \( X \) is \((0, \infty)\). By the PDF transformation formula,

\[
f_X(x) = f_U(u) \left| \frac{du}{dx} \right| = f_U(e^{-x})| - e^{-x}| = e^{-x}, \quad 0 < x < \infty
\]

Notice that \( f_U(e^{-x}) = 1 \) since \( 0 < e^{-x} \leq 1 \).

Method 2: CDF Method We have

\[
F_U(u) = \begin{cases} 
0, & x < 0 \\
u, & 0 \leq u < 1 \\
1, & u \geq 1
\end{cases}
\]

Also, by definition of a CDF,

\[
F_X(x) = P[X \leq x] = P[\ln(1/U) \leq x] = P[1/U \leq e^x] = P[U \geq e^{-x}] = 1 - F_U(e^{-x}) = 1 - e^{-x}, \quad x \geq 0.
\]

And is 0 if \( x < 0 \). So the PDF of \( X \) is

\[
f_X(x) = \frac{d}{dx} F_X(x) = e^{-x}, \quad x > 0
\]

Notice that \( X \) has an exponential distribution with rate 1.

Problem 5

This is in the text. Top of page 434.

Problem 6

Using Bayes Rule (Proposition 4.8, page 162)

\[
P[p = 1/4|X = 1] = \frac{P[X = 1|p = 1/4]P[p = 1/4]}{P[X = 1]} = \frac{P[X = 1|p = 1/4] P[p = 1/4]}{(1/4) \cdot (1/2)} = \frac{(1/4) \cdot (3/4) \cdot (1/2)}{0.25}
\]

Problem 7

We assume that the number of texts sent in each day is independent of all the other days. by the CLT, the number of texts, \( T \), sent in 30 days has approximately a normal distribution with mean \( \mu = 30 \cdot 16 = 480 \) minutes and variance \( \sigma^2 = 30 \cdot 5^2 = 750 \) minutes\(^2\). So,

\[
P[T > 500] = P \left[ Z > \frac{500 - 480}{\sqrt{750}} \right] = P[Z > 0.73] = 1 - \Phi(0.73) = 0.233.
\]
b) We want to find $N$ such that $P[T > N] \leq 0.01$. By standardizing, this is the same as

$$P\left[Z > \frac{N - 480}{\sqrt{750}}\right] = 1 - \Phi\left(\frac{N - 480}{\sqrt{750}}\right) \leq 0.01$$

$$\Rightarrow \Phi\left(\frac{N - 480}{\sqrt{750}}\right) \geq 0.99$$

Looking inside the z-table, this is only true if

$$\frac{N - 480}{\sqrt{750}} > 2.33$$

meaning we must take $N > 544$ minutes.

**Problem 8**

a) (See proposition 8.1) First check $F_X$ is non-decreasing. $-1/x$ is increasing for $x > 0$ (Think about it, $-1/1 < -1/2 < -1/3 < \text{etc...}$, so $e^{-1/x}$ is also increasing for $x > 0$ (since the function $g(z) = e^z$ is increasing). Also, $-\infty < -1/x < 0$, so $0 < e^{-1/x} < 1$, meaning $F_X$ is a nondecreasing function (try plotting it).

To check right continuity, we only need to check at $x = 0$, since it is definitely continuous everywhere else ($e^x$ is continuous for $x > 0$ and $1/x$ is continuous for $x > 0$). At $x = 0$, we have

$$\lim_{x \to 0^+} e^{-1/x} = 0$$

since $-1/x \to -\infty$ as $x \to 0^+$ and $e^{-\infty} = 0$. So $F_X$ is continuous at $x = 0$ (so it is right continuous).

Now make sure $F_X$ is 0 as $x \to -\infty$ and is 1 as $x \to \infty$. We have $\lim_{x \to -\infty} F_X(x) = 0$, since it is equal to zero for all $x < 0$.

Also, $\lim_{x \to \infty} e^{-1/x} = 1$, since $-1/x \to 0$ as $x \to \infty$, and $e^0 = 1$.

So $F_X$ is a proper CDF.

**Note:** Yeah, that got a little technical, but don’t interpret Proposition 8.1 as a list of calculus facts. Instead, go back and reread each statement in regards to the function $F_X(x) = P[X \leq x]$ (this is how CDFs are defined!). For instance, this function should be nondecreasing, since you are taking the probability of a bigger and bigger set as you increase $x$. Sorry this problem is a little calculus-heavy (which is why it’s not on the real exam).

b) To get a PDF, take a derivative of the CDF:

$$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} 0 & x \leq 0 \\ \frac{e^{-1/x}}{x^2}, & x > 0 \end{cases}$$


c) To generate random variables with given CDF, we take $x = F_X^{-1}(u)$ (This is the so-called inverse CDF method, see Proposition 8.16). So, we find $F_X^{-1}(u)$ by solving

$$u = e^{-1/x}$$

for $x$. Doing so gives

$$x = F_X^{-1}(u) = -\frac{1}{\ln(u)}$$

so take this as your random sample of $X$. 

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