1.) Consider the general damped spring-mass system

\[ mx'' + bx' + kx = 0 \]

a.) If \( m = 3, \ b = 6, \) and \( k = 15, \) find one solution \( x(t), \) and show that the corresponding energy \( E(t) = \frac{m}{2}(x')^2 + \frac{k}{2}(x)^2 \) decreases in time.

b.) Show (without finding the general solution) that for a fixed spring-mass system with any positive values of \( m, \ b, \) and \( k, \) the energy \( E(t) \) of any solution \( x(t) \) decreases in time.

2.) Consider a spring mass system with mass \( m, \) damping coefficient \( b, \) and spring constant \( k. \) Suppose we are in the underdamped case (so that \( b^2 - 4mk < 0 \)), and that \( \omega \) represents the “frequency”:

\[ \omega = \frac{\sqrt{4mk - b^2}}{2m} \]

If we replace \( m \) with a larger mass, does the new system oscillate faster or slower than the original one? (Hint: think about \( \frac{d\omega}{dm} \))

3.) Suppose we have four sinusoidally forced undamped spring-mass systems, with mass, spring constant, and forcing frequency as follows.

- A: \( m = 2, \ k = 450, \ \omega = 16 \)
- B: \( m = 4, \ k = 16, \ \omega = 4 \)
- C: \( m = 1, \ k = 121, \ \omega = 11 \)
- D: \( m = 5, \ k = 125, \ \omega = 6 \)

Identify which of the four systems exhibit resonance and which of the four exhibit beating, and match the graphs of solutions below with their corresponding systems.
4.) a.) Suppose an undamped system with mass $m$ and spring constant $k$ is externally forced by the function $f(t) = \varepsilon \cos \omega t$, where $\omega = \sqrt{k/m}$ and $\varepsilon$ is some very small positive number. Does resonance occur in this case? What does this say about the effect of the forcing amplitude on resonance?

b.) For the system with $m = 1$, $k = 25$, and $\varepsilon = .001$, find the solution with initial position $x(0) = 0$ and $x'(0) = 1$, and determine how long it takes for the amplitude to exceed that of the corresponding unforced oscillator (i.e. the solution to the same IVP except with $\varepsilon = 0$.)

5.) Consider the forced, weakly damped spring-mass system

$$x'' + .001x' + 9x = \cos(3t)$$

Notice that in absence of the damping term, this system would exhibit resonance, but since the damping term is present, no resonance occurs. Find the solution with $x(0) = 0$ and $x'(0) = 1$, and plot it. (A computer or calculator might help). Compare the plot with the plot of the corresponding solution of the undamped system $x'' + 9x = \cos(3t)$.

6.) Assume the forcing frequency $\omega$ is very close to the natural frequency $\omega_0$. Use a Taylor expansion on the solution

$$\sin \left( \frac{(\omega_0 + \omega)t}{2} \right) \sin \left( \frac{(\omega_0 - \omega)t}{2} \right)$$

to explain why this solution behaves like it resonates for a short time.