1. (20 points) Find the arc length of the curve

\[ r(t) = t^2 \mathbf{i} + 2t \mathbf{j} + \ln t \mathbf{k} \]

from the point \((1, 2, 0)\) to the point \((e^2, 2e, 1)\).

\[
\text{arc length} = \int_a^b \| r'(t) \| \, dt
\]

\[ t = 1 \text{ corresponds to } (1, 2, 0) \]
\[ t = e \text{ corresponds to } (e^2, 2e, 1) \]

\[ r'(t) = 2t \mathbf{i} + 2 \mathbf{j} + \frac{1}{t} \mathbf{k} \]

\[
\text{arc length} = \int_1^e \sqrt{4t^2 + 4 + \frac{1}{t^2}} \, dt
\]

\[
= \int_1^e \sqrt{(2t + \frac{1}{t})^2} \, dt
\]

\[
= \int_1^e (2t + \frac{1}{t}) \, dt
\]

\[
= \left[ t^2 + \ln t \right]_1^e
\]

\[
= (e^2 + 1) - 1 = e^2
\]
2. (20 points) Let \( f(x, y) = e^{2x} \sin(3x + y) \).

(a) Calculate \( \frac{\partial f}{\partial x} \).

\[
\frac{\partial f}{\partial x} = 2e^{2x} \sin(3x+y) + e^{2x} \cos(3x+y) (3) \\
= e^{2x} (2 \sin(3x+y) + 3 \cos(3x+y))
\]

(b) Calculate the gradient vector \( \nabla f(0, \pi/4) \)

\[
\frac{\partial f}{\partial y} = e^{2x} \cos(3x+y) \\
\nabla f(0, \pi/4) = (2 \frac{\sqrt{2}}{2} + 3 \frac{\sqrt{2}}{2}) \hat{e}_x + \frac{\sqrt{2}}{2} \hat{e}_y \\
= (\frac{5 \sqrt{2}}{2}) \hat{e}_x + (\frac{\sqrt{2}}{2}) \hat{e}_y
\]

(c) Calculate the directional derivative of \( f \) in the direction \( \mathbf{v} \) of most rapid increase for \( f(x, y) \) at the point \((0, \pi/4)\). Assume that \( ||\mathbf{v}|| = 1 \).

Direction of most rapid increase is direction of the gradient. Also, directional derivative = \( ||\nabla f(0, \pi/4)|| \cos \Theta \) where \( \Theta \) = angle between \( \nabla f \) and \( \mathbf{v} \). Here \( \Theta = 0 \Rightarrow \) dir derivative = \( ||\nabla f|| \)

\[
= \sqrt{\left(\frac{5}{4}\right)^2 + \left(\frac{\sqrt{2}}{4}\right)^2} = \sqrt{13}.
\]
3. (20 points) Derive the second-order Taylor polynomial of the function

\[ f(x, y) = \log(x + 2y) \]

at the point (2, 1). Express your answer in terms of \( x \) and \( y \) rather than in terms of \( h_1 \) and \( h_2 \). Note: I am using the book’s notation of \( \log \) to represent the natural logarithm.

\[ \frac{\partial f}{\partial x} = \frac{1}{x + 2y} \quad \frac{\partial f}{\partial y} = \frac{2}{x + 2y} \]

\[ \frac{\partial^2 f}{\partial x^2} = -1(x + 2y)^{-2} \quad \frac{\partial^2 f}{\partial y^2} = -2(x + 2y)^{-2} \]

\[ \frac{\partial^2 f}{\partial x \partial y} = -2(x + 2y)^{-2}. \]

At (2, 1), \( x + 2y = 4 \). \( f(2, 1) = 2 \log y \)

\[ P(x, y) = \log 4 + \left[ \frac{1}{4} \right] \left[ \begin{array}{c} x - 2 \\ y - 1 \end{array} \right] + \]

\[ -\frac{1}{2} \left[ \begin{array}{c} x - 2 \\ y - 1 \end{array} \right] \left[ \begin{array}{cc} -\frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{8} & -\frac{1}{4} \end{array} \right] \left[ \begin{array}{c} x - 2 \\ y - 1 \end{array} \right] \]

\[ P(x, y) = \log 4 + \frac{1}{4} (x-2) + \frac{1}{2} (y-1) - \frac{1}{32} (x-2)^2 - \frac{1}{8} (x-2)(y-1) - \frac{1}{8} (y-1)^2 \]
4. (20 points) Let \( P \) be the plane \( x - 2y - 2z = 1 \). Find equations for the two planes that are parallel to \( P \) and five units away from \( P \).

\[ \mathbf{n} = \text{normal vector} = \langle 1, -2, -2 \rangle \]

\[ \| \mathbf{n} \| = \sqrt{1 + 4 + 4} = 3 \]

Point on \( P \) is \((1, 0, 0)\)

Two points - one for each plane:

\((1, 0, 0) + \frac{5}{3} (1, -2, -2) = \left( \frac{8}{3}, -\frac{10}{3}, -\frac{10}{3} \right)\)

\((1, 0, 0) - \frac{5}{3} (1, -2, -2) = \left( -\frac{2}{3}, \frac{10}{3}, \frac{10}{3} \right)\)

Equation for 1 plane:

\[ (\mathbf{e} - 2\mathbf{f} - 2\mathbf{g}) \cdot ((x - \frac{8}{3})\mathbf{e} + (y + \frac{10}{3})\mathbf{f} + (z + \frac{10}{3})\mathbf{g}) = 0 \]

\[
\begin{align*}
x - \frac{8}{3} - 2(y + \frac{10}{3}) - 2(z + \frac{10}{3}) &= 0 \\
x - 2y - 2z &= \frac{48}{3} = 16
\end{align*}
\]

Equation for the other plane:

\[ (x + \frac{3}{2}) - 2(y - \frac{10}{3}) - 2(z - \frac{10}{3}) = 0 \]

\[ x - 2y - 2z = -\frac{42}{3} = -14 \]
5. (20 points) Parameterize the line of intersection of the plane tangent to \( z = e^x \sin y \) at \((0, \pi/2, 1)\) and the plane tangent to \( z = x^2 + 2y^2 \) at \((1, 2, 9)\) using a vector-valued function \( \mathbf{r}(t) \).

\[
\mathbf{n} = \left( \frac{\partial}{\partial x} \right) \mathbf{r} + \left( \frac{\partial}{\partial y} \right) \mathbf{r} - \mathbf{r}
\]

For \( z = e^x \sin y \) at \((0, \pi/2, 1)\), we get
\[
\mathbf{n}_1 = \mathbf{r} - \mathbf{r}
\]

For \( z = x^2 + 2y^2 \) at \((1, 2, 9)\), we get
\[
\mathbf{n}_2 = 2 \mathbf{i} + 8 \mathbf{j} - \mathbf{k}
\]

Line of intersection is normal to both \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \)

Direction vector \( \mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2 = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 2 & 8 & -1 \end{bmatrix} = 8 \mathbf{i} - \mathbf{j} + 8 \mathbf{k} \).

One tangent plane is \( x - (z-1) = 0 \)
\[
x - z = -1
\]

Other tangent plane is \( 2(x-1) + 8(y-2) - (z-9) = 0 \)
\[
2x + 8y - z = 9
\]

Points on the line of intersection satisfy \( x = z-1 \) and \( x + 8y = 10 \), e.g., \((2, 1, 3)\)

\[
\mathbf{r}(t) = (2+8t) \mathbf{i} + (1-t) \mathbf{j} + (3+8t) \mathbf{k}
\]