1. (14 points) Diagonalize the $2 \times 2$ matrix

$$A = \begin{bmatrix} 5 & -8 \\ 1 & -1 \end{bmatrix}.$$ 

In other words, write $A$ as $PDP^{-1}$ where $D$ is a diagonal matrix.

$$\text{char. poly } \det \begin{bmatrix} 5 - \lambda & -8 \\ 1 & -1 - \lambda \end{bmatrix} = (\lambda + 1)(\lambda - 5) + 8$$

$$= \lambda^2 - 4\lambda + 3$$

$$= (\lambda - 3)(\lambda - 1)$$

Eigenvalues: $\lambda_1 = 3$ and $\lambda_2 = 1$

$\lambda = 3 \quad \text{eigenvectors} = \text{null} \begin{bmatrix} 2 & -8 \\ 1 & -4 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\}$

$\lambda = 1 \quad \text{eigenvectors} = \text{null} \begin{bmatrix} 4 & -8 \\ 1 & -2 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

One $P = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}$, $\det P = 4 \cdot 2 = 8$

$$\Rightarrow P^{-1} = \frac{1}{8} \begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}$$
2. (14 points) In order to receive any credit, you must provide a brief justification for each answer.

(a) Suppose that a \(6 \times 9\) matrix \(A\) has four pivot columns. What are the dimensions of \(\text{Nul} \, A\), \(\text{Col} \, A\), and \(\text{Row} \, A\)?

\[
\text{four pivot cols} \Rightarrow \text{rank} \, A = 4
\Rightarrow \dim \, \text{col} \, A = \dim \, \text{row} \, A = 4
\]
\[
\text{rank} + \dim \, \text{nul} \, A = 9 \Rightarrow \dim \, \text{nul} \, A = 5.
\]

(b) What is the smallest possible dimension of the null space of a \(6 \times 9\) matrix?

\[
\text{ran} \, \text{col} + \dim \, \text{nul} \, A = 9
\text{largest rank} = 6 \Rightarrow \text{Smallest}
\]
\[
\dim \, \text{nul} \, A = 3.
\]

(c) Suppose that a \(9 \times 6\) matrix \(A\) has four pivot columns. What are the dimensions of \(\text{Nul} \, A\), \(\text{Col} \, A\), and \(\text{Row} \, A\)?

\[
\text{four pivot cols} \Rightarrow \text{rank} = 4
\Rightarrow \dim \, \text{col} \, A = \dim \, \text{row} \, A = 4
\]
\[
\text{rank} + \dim \, \text{nul} \, A = 6 \Rightarrow \dim \, \text{nul} \, A = 2.
\]

(d) What is the smallest possible dimension of the null space of a \(9 \times 6\) matrix?

\[
\text{rank} + \dim \, \text{nul} \, A = 6
\text{largest rank} = 6 \Rightarrow
\text{smallest } \dim \, \text{nul} \, A = 0.
\]
3. (16 points) Suppose

\[ v_1 = \begin{bmatrix} -1 \\ 1 \\ -3 \\ 2 \end{bmatrix} \quad v_2 = \begin{bmatrix} -2 \\ 2 \\ -3 \\ 1 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} -7 \\ 3 \\ 2 \\ 3 \end{bmatrix} \]

(a) Find the point closest to \( y \) in the subspace \( W \) spanned by \( v_1 \) and \( v_2 \).

\[ v_1 \cdot v_2 = 2 + 2 + 9 + 2 = 15 \]

Need orthogonal basis so we calculate

\[ V_3 = V_2 - \left( \frac{V_2 \cdot V_1}{V_1 \cdot V_1} \right) V_1 \quad v_1 \cdot v_1 = 15 \]

\[ V_3 = V_2 - \left( \frac{15}{15} \right) V_1 = V_2 - V_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \]

Point closest = \( \text{proj}_W y \)

\[ = \left( \frac{y \cdot V_1}{V_1 \cdot V_1} \right) V_1 + \left( \frac{y \cdot V_3}{V_2 \cdot V_2} \right) V_3 = \left( \frac{10}{15} \right) V_1 + \left( \frac{7}{3} \right) V_3 \]

\[ = \frac{2}{3} v_1 + \frac{7}{3} v_3 \]

(b) Calculate the distance of \( y \) to \( W \).

\[ \text{distance} = \| y - \text{proj}_W y \| = \| \begin{bmatrix} -4 \\ 0 \\ 4 \\ -4 \end{bmatrix} \| = 4 \| \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \| = 4 \sqrt{3} \]
4. (16 points) Calculate an orthogonal basis for the row space of the matrix

\[
A = \begin{bmatrix}
1 & -1 & 2 & -1 \\
-1 & 3 & -5 & 2 \\
2 & 2 & -2 & 0
\end{bmatrix}.
\]

Start with a basis of \( \text{Row} A - \text{row reduce} \)

\[
A = \begin{bmatrix}
1 & -1 & 2 & -1 \\
0 & 2 & -3 & 1 \\
0 & 4 & -6 & 2
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -1 & 2 & -1 \\
0 & 2 & -3 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

First two rows form a basis of \( \text{Row} A \).

Now produce an orthogonal basis

\[
V_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \quad V_1 \cdot V_2 = -9
\]

Calculate \( V_3 = V_2 - \left( \frac{V_2 \cdot V_1}{V_1 \cdot V_1} \right) V_1 \)

\[
= V_2 - \left( \frac{-9}{7} \right) \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix}
\]

\[
= \begin{bmatrix} 0 \\ 2 \\ -3 \\ 4 \end{bmatrix} + \frac{9}{7} \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{9}{7} \\ \frac{5}{7} \\ \frac{3}{7} \\ -\frac{2}{7} \end{bmatrix}
\]

Could also use

\[
V_3 = \begin{bmatrix} 9 \\ 5 \\ -3 \\ -2 \end{bmatrix}
\]

Orthogonal basis = \( \{ V_1, V_3 \} \)
5. (10 points) What can be said about a matrix $A$ that is similar to the diagonal matrix

$$D = \begin{bmatrix}
3 & 0 & 0 & 0 & 0 \\
0 & 6 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 6 \\
0 & 0 & 0 & 0 & 7
\end{bmatrix}.$$ 

Provide a brief explanation of each of your assertions.

Similar matrices have the same char poly, so the char poly of $A = (3-\lambda)^3(6-\lambda)^2(7-\lambda)$. The eigenvalues of $A$ are $\lambda = 3$ (alg mult = 3), $\lambda = 6$ (alg mult = 2), and $\lambda = 7$ (alg mult = 1).

The dim of spaces remain the same for similar matrices.

$\lambda = 3$ space of $D = \text{Span}\ \{e_1, e_3, e_4\}$

$\lambda = 6$ space of $D = \text{Span}\ \{e_2, e_5\}$

$\lambda = 7$ space of $D = \text{Span}\ \{e_6\}$.

\[ \Rightarrow \text{dim of } \lambda = 3 \text{ space of } A = 3 \]
\[ \text{dim of } \lambda = 6 \text{ space of } A = 2 \]
\[ \text{dim of } \lambda = 7 \text{ space of } A = 1 \]

Also, $\det A = \det D = (3^3)(6^2)(7) = 6804$.

Hence, $A$ is invertible as well as diagonalizable.
6. (30 points) Are the following statements true or false? **You will not receive any credit unless you justify your answers.** (Note that there are four more parts to this question on the next two pages.)

(a) If there exists a linearly-dependent set \( \{v_1, v_2, \ldots, v_p\} \) in a vector space \( V \), then \( \dim V \leq p - 1 \).

**False.** For example, \( \mathbb{R}^3 \) contains the linearly dependent set \( \{e_1, 2e_1\} \) with two vectors. However, \( \dim \mathbb{R}^3 = 3 \).

(b) Each eigenspace of a square matrix \( A \) is the null space of some matrix.

**True.** The \( \lambda \)-eigenspace is the set of all vectors \( x \) such that \( Ax = \lambda x \). If \( Ax = \lambda x \), then \( (A - \lambda I)x = 0 \Rightarrow \) the \( \lambda \)-eigenspace is \( \text{null } B \) where \( B \) is the matrix \( (A - \lambda I) \).
Question 6 (continued):

(c) Let \( W = \text{Span}\{w_1, w_2\} \). If \( v \) is orthogonal to both \( w_1 \) and \( w_2 \), then \( v \) is in \( W^\perp \).

True. Any vector \( w \) in \( W \) is a linear combination of \( w_1 \) and \( w_2 \). Compute

\[ v \cdot w = v \cdot (c_1 w_1 + c_2 w_2) = c_1 (v \cdot w_1) + c_2 (v \cdot w_2) = 0 \]

Therefore, \( v \cdot w = 0 \) for all \( w \) in \( W \).

\( \Rightarrow v \) is in \( W^\perp \).

(d) For a square matrix \( A \), an eigenvector \( v \) of \( A \) is also an eigenvector of \( A^2 \).

True. Suppose that \( Av = \lambda v \) for some scalar \( \lambda \) and non-zero \( v \).

Then \( A^2(v) = A(Av) = A(\lambda v) = \lambda (Av) = \lambda (\lambda v) = \lambda^2 v \).

So \( v \) is an eigenvector of \( A^2 \) corresponding to the eigenvalue \( \lambda^2 \) of \( A^2 \).
Question 6 (continued):

(e) For a square matrix $A$, the vectors in $\text{Col } A$ and the vectors in $\text{Nul } A$ are orthogonal.

False. $\text{Nul } A = (\text{Row } A)^\perp$.

Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$. Then $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a vector in $\text{Nul } A$. However,

$\text{Col } A = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ but

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1$. Hence, $\text{Nul } A$ and $\text{Col } A$ are not orthogonal.

(f) For a square matrix $A$, if $A^2$ is diagonalizable, then $A$ is diagonalizable.

False. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Note

that $A$ corresponds to rotation by $90^\circ$. Hence, $A$ does not have any eigenvectors. However, $A^2$ is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and this matrix is diagonal.