1 Motivation

The following is a result you can use to impress your friends at parties [and has numerous applications in PDEs and geometry]. We answer the question:

What is the volume of an \( n \)-dimensional ball with radius \( r \)? That is, for any \( x \in \mathbb{R}^n \), what is the measure of

\[
B_r(x) = \left\{ y \in \mathbb{R}^n : ||x - y||^2 = \sum_{j=1}^{n} (x_j - y_j)^2 \leq r^2 \right\}?
\]  

To answer this question, note that it suffices to compute \( B_r := B_r(0) \), since translation invariance of Lebesgue measure will give us the general result.

First, some preliminaries. What is \( I_1 = \int_{\mathbb{R}} e^{-x^2} \, dx \)? Most people automatically say \( \sqrt{\pi} \) but few can prove it.

To see how to compute this integral, we instead proceed to compute the double integral \( I_2 = \int_{\mathbb{R}^2} e^{-x^2 - y^2} \, dx \, dy \).

Using polar coordinates:

\[
I_2 = \int_{\mathbb{R}^2} e^{-x^2 - y^2} \, dx \, dy = \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2} r \, dr \, d\theta = 2\pi \cdot \frac{1}{2} e^{-r^2}|_{0}^{\infty} = \pi.
\]

Now, obviously this is a lecture on product measures, so the natural question is:

Can we conclude that \( I_2 = \int_{\mathbb{R}^2} e^{-x^2 - y^2} \, dx \, dy := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2 - y^2} \, dx \, dy = \left[ \int_{-\infty}^{\infty} e^{-x^2} \, dx \right]^2 \)?  

\[ (2) \]
The answer to this question is highly nontrivial, and is given by Tonelli’s theorem, which is proven in this lecture.

Invoking Tonelli’s theorem, we may assume that (2) is valid, so that

\[ I_2 = \left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right)^2 = \pi \Rightarrow I_1 = \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}. \]

To answer (1), we perform the integration

\[ I_n = \int_{\mathbb{R}^n} e^{-x_1^2 - x_2^2 - \cdots - x_n^2} \, dx_1 \, dx_2 \cdots \, dx_n. \]

Invoking Tonelli’s theorem, we have

\[ I_n = \left( I_1 \right)^n = \pi^{n/2}. \]

On the other hand, performing the integral in polar coordinates, we have

\[ I_n = \int_0^\infty \int_{S^{n-1}} e^{-r^2} r^{n-1} \, d\sigma \, dr = \sigma(S^{n-1}) \int_0^\infty r^{n-1} e^{-r^2} \, dr = \frac{\sigma(S^{n-1})}{2} \int_0^\infty s^{n/2 - 1} e^{-s} \, ds \]

where \( \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt \) denotes the \( \Gamma \) function, and where \( \sigma \) denotes the (induced) surface measure on \( S^{n-1} \), the boundary of an \( n \)-dimensional unit ball. The third equality results from the substitution \( s = r^2 \).

Thus, we have that the surface area of the \( n-1 \) dimensional unit sphere is given by

\[ \sigma(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma \left( \frac{n}{2} \right)}. \]

Armed with this, we now note that

\[ \mu(B_r) = \int_{\mathbb{R}^n} 1_{B_r} \, dx_1 \cdots \, dx_n = \int_0^r \int_{S^{n-1}} u^{n-1} \, d\sigma \, du = \sigma(S^{n-1}) \frac{r^n}{n} = \frac{2\pi^{n/2} r^n}{n \Gamma \left( \frac{n}{2} \right)}. \]

To compute \( \Gamma \left( \frac{n}{2} \right) \), and hence the volume of an \( n \)-dimensional ball, we give the following result:

- **Proposition 1** \( \Gamma(n+1) = n! \), and \( \Gamma \left( n + \frac{1}{2} \right) = \left( n - \frac{1}{2} \right) \left( n - \frac{3}{2} \right) \cdots \frac{1}{2} \sqrt{\pi} \) for any nonnegative integer \( n \).

While Tonelli’s theorem works with iterated integrals of nonnegative functions, Fubini’s theorem deals with iterated integrals of integrable functions. Both are related, and extremely important. We end this section by giving a brief summary of some of the most important applications of these theorems (at least in PDEs and Harmonic Analysis):

- The Fourier inversion theorem.
- For \( f, g \in L^1(\mathbb{R}^n) \), \( \hat{f} \ast g = \hat{f} \hat{g} \), where \( \hat{f} \) denotes the Fourier transform of \( f \).
2 Product Measures

Notice how many times in section 1 we needed to change the order of integration! The theorems that deal with when we can do this are the Fubini and Tonelli theorems, which we prove below.

Before we can get to them though, we need to discuss product measures.

Product measures appear in any problem where you’re dealing with a cartesian product of sets, and you want to take measures of subsets, or when you want to integrate complex valued functions defined on product spaces (as we’ve seen in section 1). It’s always healthy to keep in the back of your mind the example \( \mathbb{R}^n = \mathbb{R} \times \cdots \mathbb{R} \).

First, we make the following definitions. Throughout, we make the assumptions that \((X, \mathcal{A}, \mu)\) and \((Y, \mathcal{B}, \nu)\) are complete measure spaces.

- **Definition:** Let \( \{ X_\alpha \}_{\alpha \in I} \) is a collection of sets indexed by \( I \). The **Cartesian product** of \( \{ X_\alpha \}_{\alpha \in I} \) is \( \prod_{\alpha \in I} X_\alpha = \{ (x_\alpha) : x_\alpha \in X_\alpha \} \). When \( I = \{1, 2\} \), we write \( X_1 \times X_2 = \prod_{\alpha \in I} X_\alpha \).

- **Definition:** If \( A \subseteq X \) and \( B \subseteq Y \), we call \( A \times B \) a **rectangle**. If \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \), we call \( A \times B \) a **measurable rectangle**.

- **Definition:** We denote by \( \mathcal{R} \) the collection of all measurable rectangles.

- **Definition:** A **semialgebra** \( \mathcal{E} \) is a collection of sets such that
  1. \( \emptyset \in \mathcal{E} \),
  2. If \( E, F \in \mathcal{E} \) then \( E \cap F \in \mathcal{E} \).
  3. If \( E \in \mathcal{E} \) then \( E^c \) is a finite disjoint union of members of \( \mathcal{E} \).

Notice that \( \mathcal{R} \) is a semialgebra, since

\[
(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)
\]

and

\[
(A \times B)^c = (A^c \times B) \cup (A \times B^c) \cup (A^c \times B^c).
\]

- **Theorem 1** If \( \mathcal{E} \) is a semialgebra, the collection \( \mathcal{F} \) of finite disjoint unions of members of \( \mathcal{E} \) is an algebra.

**Proof of theorem 1:** If \( A, B \in \mathcal{E} \) then \( B^c = \bigcup_{j=1}^l C_j \) where \( \{ C_j \}_{j=1}^l \subseteq \mathcal{E} \) are disjoint. So, \( A - B = \bigcup_{j=1}^l (A \cap C_j) \) and \( A \cup B = (A - B) \cup B \), where the union is a disjoint union. So, \( A - B, A \cup B \in \mathcal{F} \).

It follows immediately by induction that if \( A_1, \ldots, A_n \in \mathcal{E} \), then \( \bigcup_{j=1}^n A_j \in \mathcal{E} \).

To see that \( \mathcal{F} \) is closed under complements, suppose \( A_1, \ldots, A_n \in \mathcal{E} \), and \( A_m^c = \bigcup_{j=1}^{J_m} B^c_j \) with
\[ B_1^1, \ldots, B_n^m \text{ disjoint members of } E. \text{ Then:} \]
\[
\left( \bigcup_{m=1}^{n} A_m \right)^c = \bigcap_{m=1}^{n} \left( \bigcup_{j=1}^{J_m} B_m^j \right) = \bigcup_{1 \leq j_m \leq J_m, 1 \leq m \leq n} B_1^1 \cap \cdots \cap B_n^n \in \mathcal{F}. \quad \square
\]

Thus, \( \mathcal{R}' \), the collection of finite disjoint unions of members of \( \mathcal{R} \) is an algebra.

- **Definition:** For a measurable rectangle \( A \times B \), we define \( \lambda(A \times B) = \mu(A) \cdot \nu(B) \).

We now verify that \( \lambda \) defines a measure on \( \mathcal{R}' \).

- **Proposition 2** Let \( \{ A_i \times B_i \} \) be a countable disjoint collection of measurable rectangles whose union is a measurable rectangle \( A \times B \). Then, \( \lambda(A \times B) = \sum_{i=1}^{\infty} \lambda(A_i \times B_i) \).

**Proof of Proposition 2:** For \( x \in X \), \( y \in Y \),
\[
\chi_A(x)\chi_B(y) = \chi_{A \times B}(x, y) = \sum_{j=1}^{\infty} \chi_{A_j \times B_j}(x, y) = \sum_{i=1}^{\infty} \chi_{A_i}(x)\chi_{B_i}(y).
\]

By the monotone convergence theorem,
\[
\mu(A)\chi_B(y) = \int_X \chi_A(x)\chi_B(y) d\mu(x) = \sum_{i=1}^{\infty} \int_X \chi_{A_i}(x)\chi_{B_i}(y) d\mu(x) = \sum_{i=1}^{\infty} \mu(A_i)\chi_{B_i}(y).
\]

In the same way,
\[
\mu(A)\nu(B) = \sum_{i=1}^{\infty} \mu(A_i)\nu(B_i) \Rightarrow \lambda(A \times B) = \sum_{i=1}^{\infty} \lambda(A_i \times B_i). \quad \square
\]

So, by the Caratheodory extension process, \( \lambda \) can be extended to a complete measure on the \( \sigma \)-algebra \( S \) generated by \( \mathcal{R}' \) containing \( \mathcal{R} \). We call this extended measure the **product measure of** \( \mu \) **and** \( \nu \) and denote it by \( \mu \times \nu \).

\[ \text{§ Remark: If } \mu \text{ and } \nu \text{ are finite (or } \sigma \)-finite) then so is \( \mu \times \nu \). Moreover, the above construction can immediately be carried over to any finite number of factors in a Cartesian product of measure spaces. Furthermore, one can show that \( S \) coincides with the product \( \sigma \)-algebra \( A \otimes B \). See the Appendix for more details.

We now prove results that will allow us to determine when we can interchange the order of integration for iterated integrals. The general idea is as follows: Fixing \( x \in X \), we define cross-sections of (finite) measurable sets \( E \) w.r.t. \( x \). This allows us to construct (measurable) functions \( g(x) \) parametrized by the transverse to the sections for which we can interchange the order of iterated integrals. Once this is done, we ramp up to nonnegative functions using the monotone convergence theorem, and general functions by applying the result to the positive and negative parts of the function. We first work with simpler measurable sets, namely measurable rectangles, and then use approximations to deal with general measurable sets.
• **Definition:** For $E \subset X \times Y$, and $x \in X$, $y \in Y$, define the **x-cross section** $E_x$ and the **y-cross section** $E^y$ of $E$ by

$$E_x = \{y \in Y : (x, y) \in E\}, \quad E^y = \{x \in X : (x, y) \in E\}.$$

• **Proposition 3** Let $x$ be a point of $X$ and $E$ be a set $\mathcal{R}_{\sigma \delta}$. Then, $E_x$ is a measurable subset of $Y$.

**Proof of Proposition 3:** If $E$ is a measurable rectangle, then $E = A \times B$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$, so that $E_x = B$ if $x \in A$ or $E_x = \emptyset$ if $x \notin A$, which in either case are obviously measurable subsets of $Y$.

Now, suppose that $E \in \mathcal{R}_{\sigma}$. Then, $E = \bigcup_{i=1}^{\infty} E_i$, where WLOG WMA that the collection $\{E_i\}$ are disjoint measurable rectangles. Then,

$$\chi_{E_x}(y) = \chi_E(x, y) = \sum_{i=1}^{\infty} \chi_{E_i}(x, y) = \sum_{i=1}^{\infty} \chi_{(E_i)_x}(y).$$

Since each $E_i$ is a measurable rectangle, each $\chi_{(E_i)_x}(y)$ is a $Y$-measurable function, so that $\chi_{E_x}$ is a $Y$-measurable function. Thus, $E_x$ is a measurable subset of $Y$.

Now suppose $E = \bigcap_{i=1}^{\infty} E_i$ with $E_i \in \mathcal{R}_{\sigma}$, and again, WLOG WMA $E_i \supset E_{i+1}$. Then, the functions $\chi_{E_i}$ are monotone decreasing. Now, $\lim_{i \to \infty} \chi_{E_i}(x) = 1$ iff $x \in \bigcap_{i=1}^{\infty} E_i$, so that

$$\lim_{i \to \infty} \chi_{E_i} = \chi_E.$$

Thus, $\chi_E$ is $Y$-measurable, so that $E$ is a measurable subset of $Y$. □

• **Proposition 4** Let $E$ be a set in $\mathcal{R}_{\sigma \delta}$ with $(\mu \times \nu)(E) < \infty$. Then, the function $g$ defined by

$$g(x) = \nu(E_x)$$

is a measurable function of $x$ and

$$\int_X g(x) d\mu(x) = (\mu \times \nu)(E).$$

**Proof of Proposition 4:** If $E$ is a measurable rectangle, then $E = A \times B$ for some $A \in \mathcal{A}$ and $B \in \mathcal{B}$. So,

$$g(x) = \nu(E_x) = \nu(B)$$

$$\Rightarrow \int_X g(x) d\mu(x) = \int_X \nu(E_x) d\mu(x) = \int_X \left( \int_Y 1_{E_x}(y) d\nu(y) \right) d\mu(x)$$

$$= \int_X \left( \int_Y 1_E(x, y) d\nu(y) \right) d\mu(x) = \int_X \left( \int_Y 1_{A \times B}(x, y) d\nu(y) \right) d\mu(x)$$

$$= \int_X \left( \int_Y 1_A(x) 1_B(y) d\nu(y) \right) d\mu(x) = \mu(A) \nu(B) = (\mu \times \nu)(E).$$
Now let $E \in \mathcal{R}_\sigma$. WLOG, WMA that $E$ is a disjoint union of measurable rectangles, $\{E_i\}_{i=1}^\infty$. Define $g_i(x) = \nu((E_i)_x)$. Then, each $g_i$ is a nonnegative measurable function, and $g = \sum_{i=1}^\infty g_i$ so that $g$ is measurable. By the monotone convergence theorem,

$$
\int_X g(x) \, d\mu(x) = \sum_{i=1}^\infty \int_X g_i(x) \, d\mu(x) = \sum_{i=1}^\infty (\mu \times \nu)(E_i) = (\mu \times \nu)(E).
$$

Now let $E \in \mathcal{R}_{\sigma\delta}$ such that $(\mu \times \nu)(E) < \infty$. Then, there are a sequence of sets $E_i \in \mathcal{R}_\sigma$ such that $E_{i+1} \subseteq E_i$ and $E = \bigcap_{i=1}^\infty E_i$. Furthermore $(\mu \times \nu)(E) < \infty$ implies that we may take $(\mu \times \nu)(E_1) < \infty$. Defining $g_i(x) = \nu((E_i)_x)$, we have that

$$
\int_X g_1(x) \, d\mu(x) = (\mu \times \nu)(E_1) < \infty
$$

so that $g_1(x) < \infty$ a.e. For some $x \in X$ such that $g_1(x) < \infty$, we have $\{(E_i)_x\}$ are a sequence of nested decreasing sets whose intersection is $E_x$, so that

$$
g(x) = \nu(E_x) = \lim_{i \to \infty} \nu((E_i)_x) = \lim_{i \to \infty} g_i(x).
$$

Thus, $g_i \to g$ a.e. so that $g$ is measurable, and by the Lebesgue Dominated Convergence Theorem,

$$
\int_X g(x) \, d\mu(x) = \lim_{i \to \infty} \int_X g_i(x) \, d\mu(x) = \lim_{i \to \infty} (\mu \times \nu)(E_i) = (\mu \times \nu)(E) \quad \Box
$$

In what follows, we use Proposition 11.6 from Royden,

- **Proposition 5** Let $\mu$ be a measure on an algebra $\mathcal{A}$, $\mu^*$ the outer measure induced by $\mu$, and $E$ be any set. Then, for $\epsilon > 0$, there is a set $A \in \mathcal{A}_\sigma$ with $E \subset A$ and

  $$
  \mu^*(A) \leq \mu^*(E) + \epsilon.
  $$

  There is also a set $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(E) = \mu^*(B)$.

- **Proposition 6** Let $E$ be a set for which $(\mu \times \nu)(E) = 0$. Then, for almost all $x$, we have $\nu(E_x) = 0$.

  **Proof of Proposition 6:** By Proposition 5, there is a set $F \in \mathcal{R}_{\sigma\delta}$ such that $E \subseteq F$ and $(\mu \times \nu)(F) = 0$. It follows from Proposition 4 that for almost all $x$, we have $\nu(F_x) = 0$. But, $E_x \subseteq F_x$ so that $\nu(E_x) = 0$ for almost all $x$, since by hypothesis $\nu$ is a complete measure on $\mathcal{B}$. $\Box$

- **Proposition 7** Let $E$ be a measurable subset of $X \times Y$ such that $(\mu \times \nu)(E)$ is finite. Then, for almost all $x$, the set $E_x$ is a measurable subset of $Y$. The function $g$ defined by

  $$
g(x) = \nu(E_x)
$$

  is a measurable function defined for almost all $x$, and

  $$
  \int X g \, d\mu = (\mu \times \nu)(E)
  $$

  6
Proof of Proposition 8: By Proposition 5, there exists $F \in \mathcal{R}_{\sigma\delta}$ such that $E \subseteq F$ and $(\mu \times \nu)(F) = (\mu \times \nu)(E)$. Defining $G = F - E$, we have that $E$ and $F$ measurable implies $G$ is measurable, so that

$$(\mu \times \nu)(F) = (\mu \times \nu)(E) + (\mu \times \nu)(G).$$

Since $(\mu \times \nu)(E) < \infty$ and equal to $(\mu \times \nu)(F)$, we have that $(\mu \times \nu)(G) = 0$. Thus, by Proposition 6, $\nu(G_x) = 0$ for almost all $x$. Hence,

$$g(x) = \nu(E_x) = \nu(F_x) \quad a.e.$$

so that $g$ is measurable by Proposition 4. Again by Proposition 4,

$$\int_X g(x) d\mu(x) = (\mu \times \nu)(F) = (\mu \times \nu)(E). \quad \square$$

• Theorem 2 (Fubini) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two complete measure spaces and $f$ be an integrable function on $X \times Y$. Then:

1. For almost all $x \in X$, the function $f_x$ defined by $f_x(y) = f(x,y)$ is an integrable function on $Y$.

2. For almost all $y \in Y$ the function $f^y$ defined by $f^y(x) = f(x,y)$ is an integrable function on $X$.

3. $\int_Y f(x,y) d\nu(y)$ is an integrable function on $X$.

4. $\int_X f(x,y) d\mu(x)$ is an integrable function on $Y$.

5. $\int_X \left[ \int_Y f d\nu \right] d\mu = \int_{X \times Y} f d(\mu \times \nu) = \int_Y \left[ \int_X f d\mu \right] d\nu.$

Proof of theorem 2: Because of the symmetry in $x$ and $y$, it suffices to prove (1), (3), and the first half of (5).

If the conclusion of the theorem holds for each of two functions, then it obviously holds for their difference, so that we may assume that $f$ is nonnegative. Proposition 8 shows that the theorem is true when $f$ is the characteristic function of a measurable set of finite measure since

$$(\mu \times \nu)(E) = \int_X g(x) d\mu(x) = \int_X \nu(E_x) d\mu(x) = \int_X \left[ \int_Y 1_{E_x}(y) d\nu(y) \right] d\mu(x)$$

and hence the theorem must be true if $f$ is a simple function that vanishes outside a set of finite measure. Now, take an increasing sequence $\{\phi_n\}$ of nonnegative simple functions such that $\phi_n \uparrow f$. Note that integrability of $f$ implies integrability of $\phi_n$ for each $n$, so that each $\phi_n$ must vanish outside a set of finite measure. Thus, $(\phi_n)_x(y) = \phi_n(x,y) \uparrow f(x,y) = f_x(y)$, so that $f_x$ is $Y$-measurable†.
By the Monotone Convergence Theorem:
\[
\int_Y f_x(y) d\nu(y) = \int_Y f(x,y) d\nu(y) = \lim_{n \to \infty} \int_Y \phi_n(x,y) d\nu(y) = \lim_{n \to \infty} \int_Y (\phi_n)_x(y) d\nu(y) \quad (*)
\]
Thus, this integral is a measurable function of \(x\).*

Now, since \(\phi_n \leq \phi_{n+1}\) we have \((\phi_n)_x \leq (\phi_{n+1})_x\). Thus,
\[
\int_Y (\phi_n)_x(y) d\nu(y) \leq \int_Y (\phi_{n+1})_x(y) d\nu(y)
\]
and so the limit in \((*)\) is a monotone limit; \(f\) integrable implying \(\phi_n\) integrable for all \(n\), and the Monotone Convergence Theorem imply:
\[
\int_X \left[ \int_Y f_x(y) d\nu(y) \right] d\mu(x) = \lim_{n \to \infty} \int_X \left[ \int_Y (\phi_n)_x(y) d\nu(y) \right] 
\]
\[
= \lim_{n \to \infty} \int_{X \times Y} \phi_n(x,y) d(\mu \times \nu)(x,y) = \int_{X \times Y} f(x,y) d(\mu \times \nu)(x,y) < \infty.
\]
Thus, \(f_x(y) d\nu(y)\) is an integrable function on \(X\), and \(\int_Y f_x(y) d\nu(y) < \infty\) a.e. \(x \in X\), so that \(f_x(y)\) is an integrable function on \(Y\) a.e. \(x \in X\). \(\square\)

**Theorem 3** (Tonelli) Let \((X,A,\mu)\) and \((Y,B,\nu)\) be two \(\sigma\)-finite measure spaces, and let \(f\) be a nonnegative measurable function on \(X \times Y\). Then:

1. For almost all \(x \in X\), the function \(f_x\) defined by \(f_x(y) = f(x,y)\) is a measurable function on \(Y\).
2. For almost all \(y \in Y\) the function \(f^y\) defined by \(f^y(x) = f(x,y)\) is a measurable function on \(X\).
3. \(\int_Y f(x,y) d\nu(y)\) is a measurable function on \(X\).
4. \(\int_X f(x,y) d\mu(x)\) is a measurable function on \(Y\)
5. \[
\int_X \left[ \int_Y f(x,y) d\nu(y) \right] d\mu(x) = \int_X \left[ \int_Y f(x,y) d\mu(x) \right] d\nu(y)
\]

**Proof of Theorem 3:** Again, by symmetry in \(x\) and \(y\), it suffices to prove (1), (3) and (5).

In theorem 2, integrability of \(f\) was only used to force the simple functions \(\phi_n\) to vanish outside a set of finite measure. If \(\mu\) and \(\nu\) are \(\sigma\)-finite, then \(\mu \times \nu\) is \(\sigma\)-finite, so that any measurable function \(f\) (not necessarily integrable!) on \(X \times Y\) is a monotone limit of simple functions, each of which vanishes outside a set of finite measure by Proposition 11.7 in Royden (why?), so that the argument in Theorem 2 still holds.

Thus, by \(\dagger\) and \(\ast\) in Theorem 2, \(f_x(y)\) is a measurable function on \(Y\) and \(\int_Y f(x,y) d\nu(y)\) is a measurable function on \(X\). (5) was shown in Theorem 2. \(\square\)
In practice, you use Tonelli before Fubini. That is, suppose you’re given a measurable function \( f : X \times Y \to \mathbb{R} \). Part of the hypothesis of Fubini is that \( f \) must be \( L^1(X \times Y) \) before we can change the order of integration of \( f \) w.r.t. \( x \) and \( y \). So, to check that \( f \) is \( L^1(X \times Y) \), we first check that \( f^+ \) and \( f^- \) are integrable, where \( f^+ \) and \( f^- \) are the usual positive and negative parts of \( f \). To do this, we use Tonelli, since for nonnegative measurable functions, we can interchange the orders of integration. Note that this is equivalent to applying Tonelli to \( |f| \).

\[ \text{§ Remarks:} \]

- Notice that in section 1, we didn’t need to use Fubini since the Gaussian function \( e^{-x^2-y^2} \) was already nonnegative.
- The proof Royden presents for Fubini and Tonelli holds for complete and \( \sigma \)-finite measure spaces. There is a version of Fubini and Tonelli for \( \sigma \)-finite measure spaces (not necessarily complete), but the proof is far less instructive. See Folland for more details.

### 2.1 Appendix

Let \( \{X_\alpha\}_{\alpha \in I} \) be an indexed collection of nonempty sets, \( X = \prod_{\alpha \in I} X_\alpha \) and \( \pi_\alpha : X \to X_\alpha \) be the projection maps defined by \( \phi_\alpha((x_\alpha)) = x_\alpha \). If \( \mathcal{M}_\alpha \) is a \( \sigma \)-algebra on \( X_\alpha \) for each \( \alpha \), the **product \( \sigma \)-algebra** on \( X \) is the \( \sigma \)-algebra generated by

\[
\{ \pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha, \ \alpha \in I \}.
\]

We denote this \( \sigma \)-algebra by \( \bigotimes_{\alpha \in I} \mathcal{M}_\alpha \). We give several propositions about this \( \sigma \)-algebra, and urge the reader to prove them.

**Proposition 8** If \( I \) is countable, then \( \bigotimes_{\alpha \in I} \mathcal{M}_\alpha \) is the \( \sigma \)-algebra generated by \( \{ \prod_{\alpha \in I} E_\alpha : E_\alpha \in \mathcal{M}_\alpha \} \).

**Proposition 9** Suppose that \( \mathcal{M}_\alpha \) is generated by \( \mathcal{E}_\alpha, \alpha \in I \). Then \( \bigotimes_{\alpha \in I} \mathcal{M}_\alpha \) is generated by \( \mathcal{F}_1 = \{ \pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{E}_\alpha, \ \alpha \in I \} \). If \( I \) is countable, and \( X_\alpha \in \mathcal{E}_\alpha \) or all \( \alpha \), \( \bigotimes_{\alpha \in I} \mathcal{M}_\alpha \) is generated by \( \mathcal{F}_2 = \{ \prod_{\alpha \in I} E_\alpha : E_\alpha \in \mathcal{E}_\alpha \} \).

**Proposition 10** Let \( X_1, \ldots, X_n \) be metric spaces, and let \( X = \prod_{j=1}^n X_j \) be equipped with the product metric. Then, \( \bigotimes_{j=1}^n B_{X_j} \subseteq B_X \). If \( X_j \) is separable for each \( j \), then \( \bigotimes_{j=1}^n B_{X_j} = B_X \).

**Proposition 11** Prove that there does not exist an infinite \( \sigma \)-algebra which has only countably many members. (Prove this one; seriously!!)