1. Does there exist an infinite $\sigma$-algebra which has only countable many members?

**Proof:** No. Suppose otherwise. Then, there exists a $\sigma$-algebra $\mathcal{F}$ on a space $X$ such that $\mathcal{F} = \{\emptyset, X, A_1, A_2, \ldots\}$.

**Claim 1.1** There exists a countably infinite subcollection $\mathcal{S} = \{A_1, A_2, \ldots\}$ of $\mathcal{F}$ such that $\bigcap_{j=1}^{\infty} A_{n_j} \neq \emptyset$.

**Proof of Claim 1.1:** Of course, we assume that $\{n_j\}_{j=1}^{\infty}$ are all distinct.

Let $n_1 = 1$. Then there must exist $A_{n_2}$ such that $A_{n_2} \cap A_{n_1} \neq \emptyset$. If not, then $A_{n_1} \cap A_j = \emptyset$ for all $j \neq n_1$. Thus, $A_{n_1} \subseteq A_{n_j}^c$ for all $j \neq n_1$.

Now, fix some $j_0 \neq n_1$ such that $A_{j_0} \neq A_{n_1}^c$. Then, since $\mathcal{F}$ is a $\sigma$-algebra, we have that $A_{k_0} = A_{j_0}^c$ for some $k_0 \neq n_1$. Thus, setting $j = k_0 \neq n_1$, $A_{n_1} \subseteq A_{k_0}^c = (A_{j_0}^c)^c = A_{j_0}$.

The only way $A_{n_1} \cap A_{j_0} = \emptyset$ and $A_{n_1} \subseteq A_{j_0}$ is if $A_{n_1} = \emptyset$, which is a contradiction.

Thus, there exists $n_2$ such that $A_{n_1} \cap A_{n_2} \neq \emptyset$.

Repeating in the manner above, we obtain sets $\{A_{n_j}\}_{j=1}^{\infty}$ such that $\bigcap_{j=1}^{\infty} A_{n_j} \neq \emptyset$. $\triangle$

Now, define $B_n = \bigcap_{j=1}^{n} A_{n_j}$. Clearly, since $\mathcal{F}$ is a $\sigma$-algebra, $B_n \in \mathcal{F}$.

It’s also clear by construction that $B_1 \supset B_2 \supset \cdots$.

Let $C_n = B_n^c \in \mathcal{F}$ (again using the property of the $\sigma$ algebra $\mathcal{F}$) so that $C_1 \subseteq C_2 \subseteq \cdots$. 
By defining $D_1 = C_1$ and $D_n = C_n - C_{n-1} \neq \emptyset$ for $n > 2$, we have that $D_n \in \mathcal{F}$ for all $n$ and $D_i \cap D_j = \emptyset$ for $i \neq j$.

Now, let $\mathcal{U}$ be the uncountable set of infinite sequences where each coordinate takes values in $\{0, 1\}$. If $u = (u_1, u_2, \ldots) \in \mathcal{U}$, define $D_u = \bigcup_{j=1}^{\infty} u_j \cdot D_j$ where we define

$$u_j \cdot D_j = \begin{cases} D_j & u_j = 1 \\ \emptyset & u_j = 0 \end{cases}$$

Since $\mathcal{F}$ is a $\sigma$-algebra, and $D_n \in \mathcal{F}$ for all $n$, it’s clear that $D_u \in \mathcal{F}$ for all $u \in \mathcal{U}$.

If $u^1, u^2 \in \mathcal{U}$ and $u^1 \neq u^2$ then $u^1_{j_0} \neq u^2_{j_0}$ for some $j_0 \geq 1$. Now, suppose that $D_{u^1} = D_{u^2}$. Then,

$$\bigcup_{j=1}^{\infty} u^1_j \cdot D_j = \bigcup_{j=1}^{\infty} u^2_j \cdot D_j.$$

Now, WLOG WMA $u^1_{j_0} = 1$ and $u^2_{j_0} = 0$. So:

$$D_{j_0} = D_{j_0} \cap \left( \bigcup_{j=1}^{\infty} u^1_j \cdot D_j \right) = D_{j_0} \cap \left( \bigcup_{j=1}^{\infty} u^2_j \cdot D_j \right) = \emptyset$$

since $\{D_j\}$ are all mutually disjoint. Since this is a contradiction, we must have that $D_{u^1} \neq D_{u^2}$.

Thus, the collection $\{D_u\}_{u \in \mathcal{U}}$ are all distinct, and belong to $\mathcal{F}$.

So, $\{D_u\}_{u \in \mathcal{U}} \subseteq \mathcal{F}$ implies that $\{D_u\}_{u \in \mathcal{U}}$ is countably infinite since $\mathcal{F}$ is countably infinite (see Lemma 7.2 of Munkres). This implies that $\mathcal{U}$ is countably infinite, which is a contradiction.

Thus, no such $\mathcal{F}$ can exist. \(\square\)

2. An uncountable sum of positive real numbers always diverges.

**Proof:** First, recall the definition of a sum over arbitrary index sets. Let $I$ be a set, and $c : I \to \mathbb{R}$.

The sum of $c$ over the set $I$ is defined by

$$\sum_{i \in I} c_i = \sup_{s \subseteq I, \ |s| < \infty} \sum_{i \in s} c_i.$$ 

Clearly, this agrees with the usual notion of sum when $I$ is countably infinite (prove it!), but generalizes this notion to uncountable index sets.

Now, define $s_0 = \{i \in I : c_i \geq 1\}$ and $s_n = \left\{i \in I : \frac{1}{n+1} > c_i \geq \frac{1}{n+1} \right\}$ for $n > 0$. If any of these sets are infinite, the sum $\sum_{i \in I} c_i$ will diverge.
To see this, note that \( i \in s \subset s_n \subset I \) where \(|s| < \infty\) implies

\[
\frac{|s|}{n+1} \leq \sum_{i \in s} c_i \leq \sup_{s \subset s_n, |s| < \infty} \sum_{i \in s} c_i \leq \sup_{s \subset I, |s| < \infty} \sum_{i \in I} c_i = \sum_{i \in I} c_i.
\]

(1)

So, if any \( s_n \) is infinite, we can make \( s \subset s_n \) s.t. \(|s| < \infty\) arbitrarily large, which makes the sum on the RHS of (1) arbitrarily large, causing it to diverge.

Thus, if \( \sum_{i \in I} c_i \) doesn’t diverge \( i.e. \sum_{i \in I} c_i = \sup_{s \subset I, |s| < \infty} \sum_{i \in s} c_i < \infty \), then each \( s_n \) must be finite. Thus, if \( \sum_{i \in I} c_i \) converges, each \( s_n \) must be finite.

If each \( s_n \) is finite, \( \bigcup_{n=0}^{\infty} s_n = \{ i \in I : c_i > 0 \} \) is countable. Thus, if \( \sum_{i \in I} c_i \) converges, then \( \{ i \in I : c_i > 0 \} \) is countable.

When \( c_i > 0 \) for all \( i \in I \), it’s clear that \( I = \{ i \in I : c_i > 0 \} \). In this case, if \( \sum_{i \in I} c_i \) converges then \( I \) is countable.

Thus, in the case of an uncountable sum of positive real numbers, \( \sum_{i \in I} c_i \) diverges. \( \square \)

**Remark:** This problem shows that in general, when a sample space \( \Omega \) is uncountably infinite, we cannot assign probabilities to individual events and expect them to be normalized to 1.

Indeed, the above proof shows that if we do so, and the probabilities are normalized, we must have that \( c_i = c(i) = 0 \) almost surely (use \( \Omega = [0, 1] \)).

### 3. For any function \( f : \Omega \to \mathbb{R} \), \( \sup_{x,y \in \Omega} |f(x) - f(y)| = \sup_{x \in \Omega} f(x) - \inf_{x \in \Omega} f(x) \)

**Proof:** First, note that

\[
f(x) - f(y) \leq |f(x) - f(y)| \leq \sup_{x,y \in \Omega} |f(x) - f(y)|
\]

\[
\Rightarrow \sup_{x \in \Omega} f(x) - \inf_{y \in \Omega} f(y) \leq \sup_{x,y \in \Omega} |f(x) - f(y)|.
\]

Now, suppose that \( x, y \in \Omega \) are fixed, and \( f(x) \geq f(y) \). Then:

\[
|f(x) - f(y)| = f(x) - f(y) \leq \sup_{z \in \Omega} f(z) - f(y) = g(y) - f(y)
\]

where \( g(y) = \sup_{z \in \Omega, f(z) \geq f(y)} f(z) \). So:

\[
|f(x) - f(y)| \leq g(y) - f(y) \leq \sup_{w \in \Omega} (g(w) - f(w)) = \sup_{w \in \Omega} g(w) - \inf_{w \in \Omega} f(w).
\]

Now, \( g(w) = \sup_{z \in \Omega, f(z) \geq f(w)} f(z) \leq \sup_{z \in \Omega} f(z) \) so that \( \sup_{w \in \Omega} g(w) \leq \sup_{z \in \Omega} f(z) \). Thus:

\[
|f(x) - f(y)| \leq \sup_{z \in \Omega} f(z) - \inf_{w \in \Omega} f(w).
\]
Since we can repeat the above with \( f(x) < f(y) \) and get the same result, we have that
\[
\sup_{x,y \in \Omega} |f(x) - f(y)| \leq \sup_{x \in \Omega} f(x) - \inf_{y \in \Omega} f(y) \Rightarrow \sup_{x,y \in \Omega} |f(x) - f(y)| = \sup_{x \in \Omega} f(x) - \inf_{y \in \Omega} f(y). \ \Box
\]

**Remark:** This gives an easy proof to the theorem: If \( \lim_{n \to \infty} x_n = x \), then \( \lim \sup_{n \to \infty} x_n = \lim \inf_{n \to \infty} x_n = x \). Let \( f(n) = x_n \). Then:
\[
\sup_{n,m \geq k} |x_n - x_m| = \sup_{n \geq k} x_n - \inf_{m \geq k} x_m.
\]

Since \( \lim_{n \to \infty} x_n = x \), \( \{x_n\} \) is Cauchy, so that given \( \epsilon > 0 \) there exists \( N \) such that \( m,n \geq N \) implies \( |x_n - x_m| < \epsilon \). Thus, \( \sup_{n,m \geq N} |x_n - x_m| < \epsilon \) so that for \( k \geq N \):
\[
\sup_{n \geq k} x_n - \inf_{n \geq k} x_n = \sup_{n \geq k} |x_n - x_m| \leq \sup_{n \geq k} |x_n - x_m| < \epsilon.
\]

Since \( \{x_n\} \) converges, \( \{x_n\} \) is bounded, so that \( \lim \sup_{n \to \infty} x_n \) and \( \lim \inf_{n \to \infty} x_n \) exist. Thus, we have that \( \lim \sup_{n \to \infty} x_n = \lim \inf_{n \to \infty} x_n \).

Finally, note that
\[
\inf_{k \geq n} x_k \leq x_n \leq \sup_{k \geq n} x_k \Rightarrow \lim \inf_{n \to \infty} x_n \leq \lim_{n \to \infty} x_n = x \leq \lim \sup_{n \to \infty} x_n \Rightarrow x = \lim \sup_{n \to \infty} x_n = \lim \inf_{n \to \infty} x_n.
\]

4. A real function \( \phi \) defined on \((a, b)\) where \(-\infty \leq a < b \leq \infty\) is convex if
\[
\phi((1 - \lambda)x + \lambda y) \leq (1 - \lambda)\phi(x) + \lambda \phi(y) \iff \frac{\phi(t) - \phi(s)}{t - s} \leq \frac{\phi(u) - \phi(t)}{u - t}
\]
whenever \( a < x < b \), \( a < y < b \), \( 0 \leq \lambda \leq 1 \) and whenever \( a < s < t < u < b \).

**Proof:** For \( a < s < t < u < b \), we may write: \( t = (1 - \lambda_t)s + \lambda_t u \) for some \( 0 < \lambda_t < 1 \). Solving for \( \lambda_t \), we have
\[
\lambda_t = \frac{t - s}{u - s}.
\]

Now,
\[
\frac{\phi(t) - \phi(s)}{t - s} \leq \frac{\phi(u) - \phi(t)}{u - t} \iff (u - t)(\phi(t) - \phi(s)) \leq (t - s)(\phi(u) - \phi(t))
\]
\[
\iff \phi(t)(u - s) \leq (u - t)\phi(s) + (t - s)\phi(u) \iff \phi(t) \leq \frac{u - t}{u - s} \phi(s) + \frac{t - s}{u - s} \phi(u)
\]
\[
\iff \phi((1 - \lambda_t)s + \lambda_t u) \leq \frac{u - \lambda_t u + \lambda_t s - s}{u - s} \phi(s) + \lambda_t \phi(u) = (1 - \lambda_t)\phi(s) + \lambda_t \phi(u)
\]
\[
\implies \frac{\phi(t) - \phi(s)}{t - s} \leq \frac{\phi(u) - \phi(t)}{u - t} \iff \phi((1 - \lambda_t)s + \lambda_t u) \leq (1 - \lambda_t)\phi(s) + \lambda_t \phi(u)
\]

We’ll be done once we prove the
Claim 1.2 If \( \frac{\phi(t) - \phi(s)}{t-s} \leq \frac{\phi(u) - \phi(t)}{u-t} \) whenever \( a < s < t < u < b \), then \( \phi \) is continuous on \((a, b)\).

**Proof of Claim 2:**

Let \( a < r < s < t < u < b \). Then, for \( x, x' \in (s, t) \), it’s clear that

\[
\frac{\phi(s) - \phi(r)}{s-r} \leq \frac{\phi(x') - \phi(x)}{x'-x} \leq \frac{\phi(u) - \phi(t)}{u-t}
\]

\[
\Rightarrow \frac{\phi(s) - \phi(r)}{s-r} \leq H(x'-x)H(\phi(x') - \phi(x)) \left| \frac{\phi(x') - \phi(x)}{x'-x} \right| \leq \frac{\phi(u) - \phi(t)}{u-t}.
\]

Thus:

\[
|\phi(x') - \phi(x)| \leq \max \left\{ \frac{\phi(u) - \phi(t)}{u-t}, -\frac{\phi(s) - \phi(r)}{s-r} \right\} |x' - x|
\]

Thus, \( \phi \) is continuous on \((s, t)\), so that \( \phi \) is continuous on \((a, b)\). \( \triangle \)

In the line before Claim 2, certainly if the RHS is true for all \( \lambda \in [0, 1] \) then it is true for all \( \lambda \in (0, 1) \) so that the LHS follows.

Conversely, the LHS implies the RHS only for \( \lambda \in (0, 1) \). But, by Claim 2, we have that \( \phi \) is continuous on \((a, b)\). Thus, using continuity of \( \phi \), the RHS must be true for all \( \lambda \in [0, 1] \). \( \square \)

5. Let \( \alpha_j \) be a sequence of nonnegative reals. Then, \( \sup \alpha_j \leq \sum_{j=1}^{\infty} \alpha_j \).

**Proof:** Clearly, if \( \sup \alpha_j = \alpha_{j_0} \) then

\[
\sup \alpha_j = \alpha_{j_0} \leq \sum_{j=1}^{\infty} \alpha_j.
\]

If the supremum is not achieved, then we may write \( \lim_{n \to \infty} \alpha_{j_n} = \sup \alpha_j \) so that since \( \alpha_{j_n} \leq \sum_{j=1}^{\infty} \alpha_j \) for all \( n \), we must have that

\[
\sup \alpha_j = \lim_{n \to \infty} \alpha_{j_n} \leq \sum_{j=1}^{\infty} \alpha_j. \quad \square
\]

6. Let \( S \) be any subset of real numbers and \( \alpha, \gamma > 0 \).

Then, \( (\sup_{s \in S} s)^\gamma = \sup_{s \in S} s^\gamma \), \( \sup_{s \in S} \alpha s = \alpha \sup_{s \in S} s \) and \( \sup_{s \in S} (s + \beta) = \sup_{s \in S} s + \beta \) for any \( \beta \in \mathbb{R} \).

**Proof:** Clearly:

\[
s \leq \sup_{s \in S} s \Rightarrow s^\gamma \leq \left( \sup_{s \in S} s \right)^\gamma \Rightarrow \sup_{s \in S} s^\gamma \leq \left( \sup_{s \in S} s \right)^\gamma
\]
\[ s \leq \sup_{s \in S} s \Rightarrow \alpha s \leq \alpha \sup_{s \in S} s \Rightarrow \sup_{s \in S} \alpha s \leq \alpha \sup_{s \in S} s \] (3)

\[ s + \beta \leq \sup_{s \in S} s + \sup_{s \in S} \beta = \sup_{s \in S} (s + \beta) \leq \sup_{s \in S} s + \beta. \] (4)

Now, if \( \sup_{s \in S} s = s_0 \), then certainly:

\[ \left( \sup_{s \in S} s \right)^\gamma = s_0^\gamma \leq \sup_{s \in S} s^\gamma. \]

If \( S \) doesn’t achieve its supremum, we may write \( \sup_{s \in S} s = \lim_{n \to \infty} s_n \) for some sequence \( \{s_n\} \) in \( S \). By continuity of the function \( f(x) = x^\gamma \), we have that \( \lim_{n \to \infty} s_n^\gamma = (\sup_{s \in S} s)^\gamma \). But, since \( s_n^\gamma \leq \sup_{s \in S} s^\gamma \) for all \( n \), we have that

\[ \left( \sup_{s \in S} s \right)^\gamma = \lim_{n \to \infty} s_n^\gamma \leq \sup_{s \in S} s^\gamma. \]

Then, \( s_n^\gamma \leq \sup_{s \in S} s^\gamma \) for all \( n \) implies that

\[ \left( \sup_{s \in S} s \right)^\gamma = \lim_{n \to \infty} s_n^\gamma \leq \sup_{s \in S} s^\gamma. \]

To finish (3) and (4) notice that:

\[ s + \beta \leq \sup_{s \in S} (s + \beta) \Rightarrow \sup_{s \in S} s + \beta \leq \sup_{s \in S} (s + \beta) \]

and

\[ \alpha s \leq \sup_{s \in S} \alpha s \Rightarrow \alpha \sup_{s \in S} s \leq \sup_{s \in S} \alpha s. \]

since \( \alpha > 0 \). Thus, the results follow. □

7. If \( f(x) \) is a continuous function, then so is \( \int_a^b f(x - y)dx \) for all intervals \([a, b]\).

**Proof:** Define

\[ F(y) = \int_a^b f(x - y)dx = \int_a^b f_y(x) \]

where \( f_y(x) = f(x - y) \). Let \( \{y_n\} \) be given such that \( y_n \to y \). Define \( f_{y_n}(x) = f(x - y_n) \). Then,

\[ F(y_n) = \int_a^b f(x - y_n)dx = \int_a^b f_{y_n}(x)dx. \]

We claim that the sequence \( \{f_{y_n}(x)\} \) is a uniformly convergent sequence of functions, with limit \( f_y(x) \).
To see this, note that for all \( x \in [a, b] \), \( |f_{y_n}(x) - f_{y}(x)| = |f(x - y_n) - f(x - y)| \); by continuity of \( f \) at \( x - y \), given \( \epsilon > 0 \) there is a \( \delta \) such that \( |x - y - w| < \delta \) implies \( |f(x - y) - f(w)| < \epsilon \). So, choosing \( N \) such that \( n \geq N \) implies \( |y - y_n| < \delta \), \( |(x - y) - (x - y_n)| < \delta \) ⇒ \( |f(x - y) - f(x - y_n)| < \epsilon \).

So, given \( \epsilon > 0 \) there exists \( N \) such that for all \( n \geq N \),

\[
|f_{y_n}(x) - f_{y}(x)| < \epsilon
\]

for all \( x \in [a, b] \). This is exactly the statement of uniform convergence of \( f_{y_n} \) to \( f_{y} \).

Thus, we may change the order of limits, so that,

\[
\lim_{n \to \infty} F(y_n) = \lim_{n \to \infty} \int_{a}^{b} f_{y_n}(x)dx = \int_{a}^{b} \lim_{n \to \infty} f_{y_n}(x)dx = \int_{a}^{b} f_{y}(x)dx = \int_{a}^{b} f(x-y)dx = F(y).
\]

Since \( \{y_n\} \) was arbitrary, as was \( [a, b] \), \( F(y) = \int_{a}^{b} f(x-y)dx \) is continuous for all intervals \([a, b]\).

\[\square\]

8. The Dirac delta function cannot be identified with any continuous function.

**Proof:** Suppose otherwise. Then, for any smooth compactly supported function \( \phi(x) \), we have

\[
\phi(0) = \int \delta(x) \phi(x)dx
\]

where \( \delta(x) \) is a continuous function. It’s clear by a change of variables:

\[
\int \delta(x-y) \phi(x)dx = \int \delta(x) \phi(y-x)dx = \phi(y)
\]

for any \( y \in \mathbb{R} \).

Now, let \( \phi_{\epsilon}(x) \) be a sequence of smooth functions supported in \([a, b]\) such that

\[
\phi_{\epsilon}(x) \to 1_{[a,b]}(x)
\]

as \( \epsilon \to 0 \) (for the existence of such a sequence, take bump functions that are 1 on \([a + \epsilon, b - \epsilon]\) and vanish outside of \((a, b)\)). Then:

\[
1_{[a,b]}(x) = \lim_{\epsilon \to 0} \phi_{\epsilon}(x) = \lim_{\epsilon \to 0} \int \delta_{y}(x) \phi_{\epsilon}(x)dx = \int_{a}^{b} \delta_{y}(x)dx
\]

by the bounded convergence theorem, where \( \delta_{y}(x) = \delta(x-y) \). By (1.7), \( 1_{[a,b]}(x) \) is continuous, which it is obviously not.

Thus, \( \delta(x) \) cannot be identified with any continuous function. \[\square\]
9. If \( f \in L^\infty \) and \( ||\tau_y f - f||_\infty \to 0 \) as \( y \to 0 \), then \( f \) agrees almost surely with a uniformly continuous function.

**Proof:** Given \( \epsilon > 0 \) there exists \( \delta \) such that \( |y| < \delta \) implies

\[
\text{ess sup}_{x \in \mathbb{R}^n} |f(x - y) - f(x)| < \epsilon \Rightarrow |f(x - y) - f(x)| < \epsilon, \text{ a.e. } x \in \mathbb{R}^n.
\]

Letting \( z = x - y \) we have that given \( \epsilon > 0 \) there exists \( \delta \) such that \( |x - z| < \delta \) implies

\[
|f(z) - f(x)| < \epsilon
\]

a.e. \( x \in \mathbb{R}^n \).

Now, let \( Z \) denote the set of measure zero for which this fails. Then, for \( z, x \in Z^c \) the above shows that \( f \) is uniformly continuous on \( Z^c \).

**Lemma 1** Given \( x \in Z \), for every \( \epsilon > 0 \) \( B(x, \epsilon) \cap Z^c \neq \emptyset \).

**Proof of Lemma:** Suppose otherwise; then, there exists \( \epsilon > 0 \) such that \( B(x, \epsilon) \cap Z^c = \emptyset \) so that \( B(x, \epsilon) \subseteq Z \Rightarrow 0 < \mu(B(x, \epsilon)) \leq \mu(Z) = 0 \) a contradiction. \( \triangle \)

By the lemma, let \( z_n \in Z^c \) such that \( z_n \to x \in Z \). Consider \( \{f(z_n)\} \). Since \( \{z_n\} \subseteq Z^c \), and \( z_n \to x \), uniform continuity of \( f \) on \( Z^c \) implies there exists \( N \) such that \( n, m > N \) implies

\[
|f(z_n) - f(z_m)| < \epsilon.
\]

Thus, \( \{f(z_n)\} \) is Cauchy so that \( f(z_n) \to L_x \). This doesn’t depend on the sequence \( \{z_n\} \) (use a shuffle sequence argument), so that we define a continuous function \( \tilde{f} \) such that \( \tilde{f}(x) = f(x) \) for \( x \in Z^c \) and \( \tilde{f}(x) = L_x \) for \( x \in Z \). Obviously, \( \tilde{f} = f \) a.e.

To show that \( \tilde{f} \) is uniformly continuous, we use the fact that \( f \) is uniformly continuous on \( Z^c \). So, given \( \epsilon > 0 \) there exists \( \delta \) such that \( |x - y| < \delta \) implies \( |f(x) - f(y)| < \epsilon \) for \( x, y \in Z^c \).

Now, consider those \( x, y \in \mathbb{R}^n \) for which \( |x - y| < \delta \). If \( x, y \in Z^c \) we’re done.

So, consider \( x \in Z \) and \( y \in B(x, \delta) \) such that \( y \in Z^c \). Since \( |f(z) - f(y)| < \epsilon \) for \( z \in B(y, \delta) \)

\[
\text{z in Z}^c \text{ letting z } \to x, \text{ we have that } |\tilde{f}(x) - \tilde{f}(y)| = |L_x - f(y)| < \epsilon.
\]

Finally, if \( x \in Z \) and \( y \in B(x, \delta) \) such that \( y \in Z \), the above shows that \( |\tilde{f}(x) - f(z)| < \epsilon \) for \( z \in B(x, \delta), z \in Z^c \). Letting \( z \to y \) we have \( |\tilde{f}(x) - \tilde{f}(y)| = |\tilde{f}(x) - L_y| < \epsilon \).

Thus, \( \tilde{f} \) is uniformly continuous, and \( f \) agrees almost surely with a uniformly continuous function. \( \square \)

10. If \( 0 < p < \infty \), put \( \gamma_p = \max\{1, 2^{p-1}\} \). Then,

\[
|\alpha - \beta|^p \leq \gamma_p(|\alpha|^p + |\beta|^p)
\]

for arbitrary complex numbers \( \alpha \) and \( \beta \).
**Proof:** Since \( f(x) = x^p \) is convex for \( p \geq 1 \), we have that for any \( x, y \in \mathbb{R} \), \( x, y \geq 0 \):

\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).
\]

So, letting \( t = \frac{1}{2} \):

\[
\left( \frac{1}{2}x \right)^p + \left( \frac{1}{2}y \right)^p \leq \frac{1}{2}x^p + \frac{1}{2}y^p
\]

\[
\Rightarrow (x + y)^p \leq 2^{p-1}(x^p + y^p).
\]

So, for \( \alpha, \beta \in \mathbb{C} \):

\[
|\alpha - \beta|^p \leq (|\alpha| + |\beta|)^p \leq 2^{p-1}(|\alpha| + |\beta|) = \gamma_p(|\alpha|^p + |\beta|^p)
\]

since obviously for \( p \geq 1 \), \( \gamma_p = \max\{1, 2^{p-1}\} = 2^{p-1} \).

Now, for \( 0 < p < 1 \), fix \( \alpha \in \mathbb{C} \) and consider the function

\[
f(x) = |\alpha|^p + x^p - (|\alpha| + x)^p
\]

for \( x \geq 0 \). Then, for \( x > 0 \):

\[
f'(x) = px^{p-1} - p(|\alpha| + x)^{p-1}.
\]

But, since \( 0 < p < 1 \):

\[
|\alpha| + x \geq x \Leftrightarrow (|\alpha| + x)^{1-p} \geq x^{1-p} \Leftrightarrow x^{p-1} \geq (|\alpha| + x)^{p-1} \Leftrightarrow x^{p-1} - (|\alpha| + x)^{p-1} \geq 0
\]

\[
\Leftrightarrow f'(x) = p(x^{p-1} - (|\alpha| + x)^{p-1}) \geq 0.
\]

Thus, \( f(x) \) is increasing for \( x > 0 \). Since \( f(0) = 0 \), we must therefore have \( f(x) \geq 0 \) for all \( x \geq 0 \)

\[
\Rightarrow |\alpha|^p + x^p - (|\alpha| + x)^p \geq 0 \Leftrightarrow (|\alpha| + x)^p \leq |\alpha|^p + x^p.
\]

Thus, for any complex number \( \beta \in \mathbb{C} \),

\[
|\alpha - \beta|^p \leq (|\alpha| + |\beta|)^p \leq |\alpha|^p + |\beta|^p = \gamma_p(|\alpha|^p + |\beta|^p)
\]

since obviously for \( 0 < p < 1 \), \( \gamma_p = \max\{1, 2^{p-1}\} = 1 \). □

11. \( \sum_{n=1}^{\infty} \rho^{n^s} \leq \frac{\Gamma\left(\frac{1}{s}\right)}{s|\ln|\rho|\|^s} < \infty \) for all \( s > 0 \) and \( |\rho| < 1 \).

**Proof:** WLOG, WMA \( 0 < \rho < 1 \). We employ the integral test, and hence, consider

\[
\sum_{n=1}^{\infty} \rho^{n^s} \leq \int_{0}^{\infty} \rho^x s^{x^s} dx = \int_{0}^{\infty} e^{-x|\ln|\rho||} dx.
\]
Now, let $u = x^s \ln \rho$. Then,
\[ du = sx^{s-1} \ln \rho \, dx = \frac{du}{sx^{s-1} \ln \rho} = \frac{x^{1-s}du}{s} = \frac{u^{\frac{1}{s}-1}}{s} \ln \rho \]
so that
\[ \sum_{n=1}^{\infty} \rho_n \leq \int_{0}^{\infty} \rho^s \, dx = \int_{0}^{\infty} e^{-x} \ln \rho \, dx = \frac{1}{s} \ln \rho \int_{0}^{\infty} u^{\frac{1}{s}-1}e^{-u}du = \frac{\Gamma\left(\frac{1}{s}\right)}{s \ln \rho^{\frac{1}{s}}}. \]

For $s > 0$, $\Gamma\left(\frac{1}{s}\right) < \infty$ so that for $s > 0$, $\int_{0}^{\infty} \rho^s \, dx < \infty$. Thus, for all $s > 0$ and any $|\rho| < 1$, $\sum_{n=1}^{\infty} \rho_n < \infty \quad \Box$.

12. (The Sobolev Lemma) If $s > k + \frac{1}{2}n$ then $H_s \subseteq C^k$ and there is a constant $C = C_{s,k}$ such that
\[ \sup_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} |\partial^\alpha f(x)| \leq C||f||_s. \tag{5} \]

Furthermore for $|\alpha| = k$, $\partial^\alpha f$ is uniformly Holder continuous with exponent $s - k$.

**Proof:** Recall that $H_s = H_s(\mathbb{R}^n) = \{ f \in S'(\mathbb{R}^n) : \hat{f}$ is a function and $||f||_2^2 = \int |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi < \infty \}$. Now, the distributional derivative $\partial^\alpha \hat{f}(\xi)$ is given by the computation:
\[ \langle \partial^\alpha \hat{f}(\xi), \phi(\xi) \rangle = (-1)^{|\alpha|} \langle f(x), \partial^\alpha \hat{\phi}(x) \rangle = (-1)^{|\alpha|} \langle f(x), (-2\pi i \xi)^\alpha \hat{\phi}(x) \rangle = \langle (2\pi i\xi)^\alpha \hat{f}(\xi), \phi(\xi) \rangle \]
so that $\partial^\alpha \hat{f}(\xi) = (2\pi i\xi)^\alpha \hat{f}(\xi)$. Thus, for $|\alpha| \leq k$:
\[ ||\partial^\alpha \hat{f}(\xi)||_1 = \int \,(2\pi i\xi)^\alpha \hat{f}(\xi) |d\xi| \leq (2\pi)^k \int (1 + |\xi|^2)^{\frac{k}{2}} |\hat{f}(\xi)| |d\xi| = (2\pi)^k \int (1 + |\xi|^2)^{\frac{k}{2}} |\hat{f}(\xi)|(1 + |\xi|^2)^{\frac{k-s}{2}} |d\xi| \]
\[ \leq (2\pi)^k \left[ \int (1 + |\xi|^2)^{s} |\hat{f}(\xi)|^2 d\xi \int (1 + |\xi|^2)^{s-k} d\xi \right]^{\frac{1}{2}}. \]
Since $f \in H_s$, the first term in the last expression is finite. For the second term, notice that $s > k + \frac{1}{2}n$ implies:
\[ \int (1 + |\xi|^2)^{s-k} d\xi = \omega_n \int_{0}^{\infty} (1 + r^2)^{s-k} r^{n-1}dr < \infty. \]

Thus, $||\partial^\alpha \hat{f}(\xi)||_1 < \infty$ for all $|\alpha| \leq k$ so that by the Fourier inversion theorem for tempered distributions, $\partial^\alpha f(x)$ is continuous, and
\[ \sup_{x \in \mathbb{R}^n} |\partial^\alpha f(x)| \leq ||\partial^\alpha \hat{f}(\xi)||_1 \leq C||f||_s. \]
Since this is true for all $|\alpha| \leq k$, (5) follows. By #12, this implies that $f \in C^k$. 

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To prove the Holder continuity, assume that $k$ is the largest integer for which the result holds. Then, we may assume: $k + \frac{1}{2} n < s \leq k + \frac{1}{2} n + 1$ so that $0 < s - k - \frac{1}{2} n \leq 1$. From above:

$$
\partial^k f(x) = \int \partial^\alpha f(\xi) e^{2\pi i \xi \cdot x} d\xi = \int (2\pi i \xi)^k f(\xi) e^{2\pi i \xi \cdot x} d\xi 
$$

$$
\Rightarrow \left| \frac{\partial^k f(x) - \partial^k f(y)}{|x - y|^{s - k - \frac{1}{2} n}} \right| \leq (2\pi)^k \int |\xi|^k |f(\xi)| \left| \frac{e^{2\pi i \xi \cdot x} - e^{2\pi i \xi \cdot y}}{|x - y|^{s - k - \frac{1}{2} n}} \right| d\xi
$$

(6)

Now, for $|x - y|^{-1} \leq |\xi|:

$$
\left| \frac{e^{2\pi i \xi \cdot x} - e^{2\pi i \xi \cdot y}}{|x - y|^{s - k - \frac{1}{2} n}} \right| \leq \frac{2}{|x - y|^{s - k - \frac{1}{2} n}} \leq 2|\xi|^{s - k - \frac{1}{2} n}.
$$

For $|x - y|^{-1} \geq |\xi|:

$$
\left| \frac{e^{2\pi i \xi \cdot x} - e^{2\pi i \xi \cdot y}}{|x - y|^{s - k - \frac{1}{2} n}} \right| \leq \frac{|\xi||x - y|}{|x - y|^{s - k - \frac{1}{2} n}} \leq |\xi|^{s - k - \frac{1}{2} n}.
$$

So splitting the integral on the RHS of (6) into two pieces for $|\xi| < |x - y|^{-1}$ and $|\xi| \geq |x - y|^{-1}$:

$$
\left| \frac{\partial^k f(x) - \partial^k f(y)}{|x - y|^{s - k - \frac{1}{2} n}} \right| \leq (2\pi)^k \int 2|\xi|^k |f(\xi)||\xi|^{s - k - \frac{1}{2} n} d\xi = K < \infty
$$

13. Give an example of a series such that $\sum_{n=1}^{\infty} a_n$ converges absolutely, but $a_n \notin O \left( \frac{1}{n^{1+\delta}} \right)$ for any $\delta > 0$.

**Proof:** Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^{1+\log n}}$. This series converges absolutely (Rudin, pg. 63) but by L’Hospital’s rule, $a_n \notin O \left( \frac{1}{n^{1+\delta}} \right)$ for any $\delta > 0$. □

14. Is a function that is almost surely continuous on a compact set bounded almost surely?

**Proof:** No. Look at $f(x) = \tan x$ for $x \in \left[ 0, \frac{\pi}{2} \right]$ and $f \left( \frac{\pi}{2} \right) = 0$. Obviously, this is continuous almost surely on $\left[ 0, \frac{\pi}{2} \right]$ but isn’t bounded. □

15. All norms on $\mathbb{R}^n$ ($\mathbb{C}^n$) are equivalent

**Proof:** The proof for $\mathbb{C}^n$ is the same, so we show the result for $\mathbb{R}^n$. Let $| \cdot |$ be the Euclidean norm and $|| \cdot ||$ be any norm.

**Step 1:** $|| \cdot || : \mathbb{R}^n \to \mathbb{R}$ is continuous w.r.t. $| \cdot |$.

To see this, note that if $x = \sum_{j=1}^{n} a_j e_j$ then

$$
||x|| = \left| \sum_{j=1}^{n} a_j e_j \right| \leq \sum_{j=1}^{n} |a_j|||e_j|| \leq \sup_{1 \leq j \leq n} ||e_j|| \sum_{j=1}^{n} |a_j| = M \langle \alpha, 1 \rangle
$$
where \( M = \sup_{1 \leq j \leq n} ||e_j|| \), \( \alpha = (|a_1|, |a_2|, \ldots, |a_n|) \), 1 = (1, \ldots, 1) \( n \)-times and \( \langle \cdot, \cdot \rangle \) is the Euclidean inner product.

Clearly, \( |\langle \alpha, 1 \rangle| = \langle \alpha, 1 \rangle \) so that by the Cauchy-Schwartz inequality:

\[
\langle \alpha, 1 \rangle \leq ||\alpha||1 = ||\alpha||\sqrt{n}.
\]

But, \( ||\alpha|| = \sqrt{\sum_{j=1}^{n} |a_j|^2} = ||x|| \). Thus:

\[
||x|| \leq M \langle \alpha, 1 \rangle \leq M ||\alpha||\sqrt{n} = M \sqrt{n} ||x||.
\]

This implies that when \( |x - x_0| < \delta \), \( ||x - x_0|| \leq M \sqrt{n} |x - x_0| < M \sqrt{n} \delta = \epsilon \). So, given \( \epsilon > 0 \), choose \( \delta = \frac{\epsilon}{M \sqrt{n}} \). Thus, \( || \cdot || \) is continuous w.r.t. \( | \cdot | \).

**Step 2:** \( \alpha||x||_a \leq ||x||_b \leq \beta||x||_a \forall x \in S^{n-1} \), where \( \alpha, \beta > 0 \) and where \( S^{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \} \).

Consider the function \( f : \mathbb{R}^n - \{0\} \rightarrow \mathbb{R} \) defined by \( f(x) = \frac{||x||_b}{||x||_a} \). By Step 1, this function is continuous on \( \mathbb{R}^n - \{0\} \). Consider its restriction to \( S^{n-1} \), which is compact in the topology induced by \( | \cdot | \). Then, \( f \) is a continuous function on a compact set, and so must achieve its minimum and maximum; that is, \( \alpha \leq f(x) \leq \beta \forall x \in S^{n-1} \) implying that

\[
\forall x \in S^{n-1}. \text{ Since the norms are positive, } \alpha, \beta > 0. \text{ Thus: } \alpha||x||_a \leq ||x||_b \leq \beta||x||_a \forall x \in S^{n-1}.
\]

**Step 3:** \( \alpha||x||_a \leq ||x||_b \leq \beta||x||_a \forall x \in \mathbb{R}^n \).

Given \( x \in \mathbb{R}^n, x = |x|\hat{x} \) where \( \hat{x} \) is a unit vector in \( \mathbb{R}^n \) w.r.t. \( | \cdot | \). Thus, \( \hat{x} \in S^{n-1} \), and so by Step 2:

\[
\alpha||\hat{x}||_a \leq ||\hat{x}||_b \leq \beta||\hat{x}||_a \Rightarrow \alpha|x|||\hat{x}||_a \leq |x|||\hat{x}||_b \leq \beta|x|||\hat{x}||_a
\]

\[
\Rightarrow \alpha||x||_a \leq ||x||_b \leq \beta||x||_a.
\]

Since the norms \( || \cdot ||_a \) and \( || \cdot ||_b \) were arbitrary, as was \( x \in \mathbb{R}^n \) the result follows. \( \square \)

16. \( \det e^A = e^{trA} \) for any \( A \in M_{m,m}(\mathbb{C}) \).

**Proof:** Note that the proof that \( \sum_{j=1}^{\infty} \frac{A^j}{j!} = e^A = \lim_{n \rightarrow \infty} (1 + \frac{A}{n})^n \) when \( A \in \mathbb{C} \) carries over to when \( A \) is an \( m \times m \) matrix.

We have:

\[
\det \left( I + \frac{A}{n} \right)^n = \left[ \det \left( I + \frac{A}{n} \right) \right]^n = \prod_i \left( 1 + \frac{\lambda_i}{n} \right)^n = \prod_i \left( 1 + \frac{\lambda_i}{n} \right)^n
\]
where \( \{\lambda_i\} \) are the eigenvalues of \( A \) counted with multiplicity.

Now, the determinant function is a continuous function on \( M_{m,m}(\mathbb{C}) \) in any norm. To see this, we consider \( M_{m,m}(\mathbb{C}) \) as a vector space isomorphic to \( \mathbb{C}^{n^2} \). Since any two norms on \( \mathbb{C}^{n^2} \) are equivalent by problem 15, any two norms on \( M_{m,m}(\mathbb{C}) \) are equivalent. Thus, if \( A_n \to A \) in any norm, we must have that \( A_n(i,j) \to A(i,j) \) as \( n \to \infty \). Since \( \det A_n = P(A_n(i,j)) \) for some polynomial \( P \) and polynomials are continuous functions, we must have

\[
\det A_n \to \det A \quad \text{as} \quad A_n \to A
\]

in any norm.

Thus:

\[
\det e^A = \det \left[ \lim_{n \to \infty} \left( I + \frac{A}{n} \right)^n \right] = \lim_{n \to \infty} \det \left[ \left( I + \frac{A}{n} \right)^n \right] = \lim_{n \to \infty} \prod_i \left( 1 + \frac{\lambda_i}{n} \right)^n = \prod_i e^{\lambda_i} = e^{\text{tr}A}. \quad \square
\]

17. There are uncountably many distinct proper subspaces in an infinite dimensional separable Hilbert space.

**Proof:** Let \( \mathcal{H} \) be an infinite dimensional separable Hilbert space. Let \( \{\phi_k\} \) be an orthonormal basis for \( \mathcal{H} \). Let \( S \) be the set of all subsets of \( \mathbb{N} \). By classical set theory, \( S \) is uncountable. But, each set in \( S \) corresponds to a unique subcollection of \( \{\phi_k\} \) which upon taking its span corresponds to a unique finite dimensional subspace of \( \mathcal{H} \). Since there are uncountably many such subspaces, the number of distinct proper subspaces in \( \mathcal{H} \) is uncountable. \( \square \)

18. \[ \sum_{|\alpha|=0}^{\infty} x^\alpha = \frac{1}{\prod_{i=1}^{n-1} (1-x_i)} \] where \( x = (x_1, \ldots, x_n) \) and \( |x_i| < 1 \) for \( 1 \leq i \leq n \).

**Proof:** Proceed by induction on the length of the multinomial \( x \). If \( n = 1 \), obviously the result is true. So, suppose that

\[
\sum_{|\alpha|=0}^{\infty} x^\alpha = \frac{1}{\prod_{i=1}^{n-1} (1-x_i)}
\]

for \( x = (x_1, \ldots, x_{n-1}) \).

For the multinomial of length \( n \), writing \( \beta = (\alpha_1, \ldots, \alpha_{n-1}) \) and \( y^\beta = x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} \):

\[
\sum_{|\alpha|=0}^{K} x^\alpha = \sum_{j=0}^{K} \left( \sum_{|\alpha|=j} x^\alpha \right) = \sum_{j=0}^{K} \left( \sum_{|\beta|+\alpha_n=j} y^\beta \cdot x_n^{\alpha_n} \right) = \sum_{j=0}^{K} \left( \sum_{\alpha_n=0}^{j} \sum_{|\beta|=j-\alpha_n} y^\beta \cdot x_n^{\alpha_n} \right)
\]
\[ = \sum_{j=0}^{K} \sum_{\alpha_n=0}^{j} x^\alpha_n \sum_{|\beta|=j-\alpha_n} y^\beta = \left( \sum_{|\beta|=0} y^\beta \right) + \left( \sum_{|\beta|=1} y^\beta + x_n \sum_{|\beta|=0} y^\beta \right) + \left( \sum_{|\beta|=2} y^\beta + x_n \sum_{|\beta|=1} y^\beta + x_n^2 \sum_{|\beta|=0} y^\beta \right) + \cdots + \left( \sum_{|\beta|=K} y^\beta + x_n \sum_{|\beta|=K-1} y^\beta + \cdots + x_n^K \sum_{|\beta|=0} y^\beta \right) \]
\[ = \sum_{j=0}^{K} \sum_{|\beta|=j} y^\beta + x_n \sum_{j=0}^{K-1} \sum_{|\beta|=j} y^\beta + \cdots + x_n^K \]

so that taking the limit \( K \to \infty \) and recalling that \( |x_i| < 1 \) for all \( i \):
\[ \sum_{|\alpha|=0}^{\infty} x^\alpha = \sum_{j=0}^{\infty} \sum_{|\beta|=j} y^\beta + x_n \sum_{j=0}^{\infty} \sum_{|\beta|=j} y^\beta + \cdots = \frac{1}{\prod_{i=1}^{n-1} (1 - x_i)} \sum_{j=0}^{\infty} x^j = \frac{1}{\prod_{i=1}^{n-1} (1 - x_i)}. \]

19. For an \( n \times n \) matrix, show that \( \det A = \det A' \).

**Proof:** Suppose that \( \lambda \) is an eigenvalue of \( A \). Then, \( \det(A - \lambda I) = 0 \), so that \( A - \lambda I \) is not invertible, and so \( \ker(A - \lambda I) \neq \{0\} \). By Rank-Nullity, this implies that \( \text{Im}(A - \lambda I) \subseteq \mathbb{R}^n \) so that \( \text{Im}(A - \lambda I)^\perp \neq \{0\} \) and \( \dim \ker(A - \lambda I) = \dim \text{Im}(A - \lambda I)^\perp \). Since \( \ker((A - \lambda I)^\prime) = \text{Im}(A - \lambda I)^\perp \) we have that \( \ker((A - \lambda I)^\prime) \neq \{0\} \) and \( \dim \ker((A - \lambda I)^\prime) = \dim \ker(A - \lambda I) \) so that \( \det(A' - \lambda I) = 0 \). Thus, \( \lambda \) is an eigenvalue of \( A' \) and \( \dim E^A_\lambda = \dim E^A_\lambda' \).

Passing to generalized eigenvectors and working over \( \mathbb{C} \) if necessary, we conclude that \( A \) and \( A' \) have the same Jordan Canonical form, and hence must have the same determinants. \( \square \)
2 Stochastic Processes

1. (Kolmogorov) Let $X_t, \ t \in [0, 1)^d$ be a Banach-valued process for which there exists three strictly positive constants $\gamma, c, \epsilon$ such that

$$E \left[ |X_t - X_s|^{\gamma} \right] \leq c|t - s|^{d+\epsilon}.$$ 

Then, there is a modification $\tilde{X}$ of $X$ such that

$$E \left[ \left( \sup_{s \neq t} \frac{|\tilde{X}_t - \tilde{X}_s|}{|t - s|^\alpha} \right)^\gamma \right] < \infty$$

for every $\alpha \in \left[ 0, \frac{\epsilon}{\gamma} \right)$. In particular, the paths of $\tilde{X}$ are Hölder continuous of order $\alpha$.

**Proof:** In this problem only, we assume that $|t| = \sup_{i} t_i$ for any vector $t \in \mathbb{R}^d$.

For $m \in \mathbb{N}$, define

$$D_m = \{(2^{-m}i_1, \ldots, 2^{-m}i_d) : i_k \in [0, 2^m), \ i_k \in \mathbb{Z}, \ \forall 1 \leq k \leq d\}.$$ 

Set $D = \bigcup_{m=1}^{\infty} D_m$, and define $\Delta_m = \{(s, t) \in D \times D_m : |s - t| = 2^{-m}\}$.

Notice that $|\Delta_m| < 2^{md}$, since there are a total of $2^{md}$ possible choices for both $s$ and $t$. For $s, t \in D$, we say that $s \leq t$ if each component of $s$ is less than or equal to the corresponding component of $t$.

Now define $K_i = \sup_{(s, t) \in \Delta_i} |X_s - X_t|$. By hypothesis:

$$E[K_i^\gamma] = E \left[ \left( \sup_{(s, t) \in \Delta_i} |X_s - X_t| \right)^\gamma \right] = E \left[ \left( \sup_{(s, t) \in \Delta_i} |X_s - X_t|^{\gamma} \right) \right] \leq \sum_{(s, t) \in \Delta_i} E \left[ |X_s - X_t|^{\gamma} \right]$$

$$\leq 2^{id} \cdot c 2^{-i(d+\epsilon)} = c 2^{-i\epsilon}.$$ 

Now, since $D$ is the dyadic unit cube in $\mathbb{R}^d$, for any $s \in D$, there is an increasing sequence $s_n$ of points in $D$ such that $s_n \in D_n, s_n \leq s$, and $s_n = s$ for all $n$ sufficiently large.

Now, let $s, t \in D$ and $|s - t| \leq 2^{-m}$. Let $s_m$ and $t_m$ be sequences in $D$ of the type described above. Then,

$$X_s - X_t = \sum_{i=m}^{\infty} (X_{s_{i+1}} - X_{s_i}) + X_{s_m} - X_{t_m} + \sum_{i=m}^{\infty} (X_{t_i} - X_{t_{i+1}})$$

where the series are actually finite sums, since $t_n = t$ and $s_m = s$ for $n$ and $m$ sufficiently large.
Thus:

\[ |X_s - X_t| \leq \sum_{i=m}^{\infty} |X_{s_{i+1}} - X_{s_i}| + |X_{s_m} - X_{t_m}| + \sum_{i=m}^{\infty} |X_{t_i} - X_{t_{i+1}}| \leq 2 \sum_{i=m}^{\infty} K_{i+1} + K_m \leq 2 \sum_{i=m}^{\infty} K_i. \]

Now, set

\[ M_\alpha = \sup_{s,t \in D, s \neq t} \left\{ \frac{|X_t - X_s|}{|t - s|^\alpha} \right\}. \]

Notice that for any \( s, t \in D, s \neq t \), we can always find an \( m \) large enough such that

\[ 2^{-m} \leq |t - s| \leq 2^{-m}. \]

Thus, for any \( s, t \in D \) such that \( s \neq t \),

\[ \frac{|X_t - X_s|}{|t - s|^\alpha} \leq 2^{\alpha(m+1)} \sup_{|t - s| \leq 2^{-m}} |X_t - X_s| \]

\[ \leq \sup_{m \in \mathbb{N}} \left\{ 2^{(m+1)\alpha} \sup_{|t - s| \leq 2^{-m}} |X_t - X_s| \right\} \]

\[ \Rightarrow M_\alpha = \sup_{s,t \in D, s \neq t} \left\{ \frac{|X_t - X_s|}{|t - s|^\alpha} \right\} \leq \sup_{m \in \mathbb{N}} \left\{ 2^{(m+1)\alpha} \sup_{|t - s| \leq 2^{-m}} |X_t - X_s| \right\} \]

\[ \leq \sup_{m \in \mathbb{N}} \left\{ 2 \cdot 2^{(m+1)\alpha} \sum_{i=m}^{\infty} K_i \right\} \leq 2^{\alpha+1} \sum_{i=0}^{\infty} 2^{i\alpha} K_i. \]

So, for \( \gamma \geq 1 \) and \( \alpha < \frac{\xi}{\gamma} \), we get

\[ \|M_\alpha\|_\gamma \leq 2^{\alpha+1} \sum_{i=0}^{\infty} 2^{i\alpha} \|K_i\|_\gamma \leq 2^{\alpha+1} \sum_{i=0}^{\infty} 2^{i\alpha} \cdot (c2^{-i})^{\frac{1}{\gamma}} = J \sum_{i=0}^{\infty} 2^{i\left(\frac{\alpha}{\gamma} - \frac{1}{\gamma}\right)} < \infty \]

where \( J = 2^{\alpha+1} c^{\frac{1}{\gamma}} \).

For \( \gamma < 1 \) we use (1.10), so that

\[ E \left[ (M_\alpha)^\gamma \right] \leq 2^{\gamma(\alpha+1)} \sum_{i=0}^{\infty} 2^{i\gamma} E [K_i]^\gamma = 2^{\gamma(\alpha+1)} \sum_{i=0}^{\infty} 2^{i\gamma} c2^{-i} = J' \sum_{i=0}^{\infty} 2^{i\gamma(\alpha - \frac{1}{\gamma})} < \infty \]

where \( J' = c2^{\gamma(\alpha+1)} \).

Thus, for \( \alpha < \frac{\xi}{\gamma} \) we have:

\[ M_\alpha = \sup_{s,t \in D, s \neq t} \left\{ \frac{|X_t - X_s|}{|t - s|^\alpha} \right\} < \infty, \ a.e. \]

so that for almost all \( \omega \), \( X \) is uniformly continuous on \( D \). Thus, we may define

\[ \tilde{X}_t(\omega) = \lim_{s \to t, s \in D} X_s(\omega) \]
for almost all \( \omega \).

By Fatou’s lemma,
\[
E \left[ |\tilde{X}_t - X_t| \right] \leq \liminf_{s \to t, \ s \in D} E \left[ |\tilde{X}_s - X_s| \right] = 0
\]
so that \( \tilde{X}_t = X_t \) a.e. Thus, \( \tilde{X} \) is a modification of \( X \) whose sample paths are Hölder continuous with exponent \( \alpha \).

\[ \square \]

2. Suppose \( H \) is an infinite dimensional Hilbert space and \( m \) a countably additive measure on \( B_H \) which is invariant under translations and satisfies \( m(B(0, \epsilon)) > 0 \) for all \( \epsilon > 0 \). Show that \( m(V) = \infty \) for all nonempty open subsets \( V \subset H \).

**Proof:** WLOG, WMA that \( H \) is separable, since in the inseparable case, one only needs to restrict to a countable subset of basis vectors.

Let \( \epsilon > 0 \) be given and consider \( B(0, \epsilon) \). Let \( \{e_1, \ldots\} \) be a complete orthonormal system for \( H \).

Let \( R = \frac{\epsilon}{1 + \sqrt{2}} \) and \( \delta = \frac{\epsilon}{1 + \sqrt{2}} \).

Clearly, \( B(Re_j, \delta) \subset B(0, \epsilon) \) for each \( j \). We shall show that all the balls \( B(Re_j, \delta) \) are disjoint.

Suppose that \( x \in B(Re_j, \delta) \cap B(Re_k, \delta) \) for \( j \neq k \). Then:
\[
\frac{2\epsilon}{\sqrt{2} + 1} = \sqrt{2}R = ||Re_j - Re_k|| \leq ||Re_j - x|| + ||x - Re_k|| < 2\delta = \frac{2\epsilon}{\sqrt{2} + 1}
\]
which is a contradiction. Thus, \( B(Re_j, \delta) \cap B(Re_k, \delta) = \emptyset \) so that since \( j \neq k \) were arbitrary, the balls \( B(Re_j, \delta) \) are disjoint.

Clearly, there are infinitely many \( B(Re_j, \delta) \). Since \( m \) is translation invariant and \( m(B(0, \eta)) > 0 \) for all \( \eta > 0 \), \( m(B(Re_j, \delta)) = m(B(0, \delta)) = M > 0 \). Since the balls \( B(Re_j, \delta) \) are all mutually disjoint, countable additivity of \( m \) implies
\[
\infty = \sum_{j=1}^{\infty} M = \sum_{j=1}^{\infty} m(B(Re_j, \delta)) = m \left( \bigcup_{j=1}^{\infty} B(Re_j, \delta) \right) \leq B(0, \epsilon).
\]

Since \( \epsilon > 0 \) was arbitrary, \( m(B(0, \epsilon)) = \infty \) for all \( \epsilon > 0 \). Since any open subset \( V \subset H \) contains some ball of radius \( \epsilon > 0 \), translation invariance again implies that \( m(V) = \infty \).

**Remark:** In particular, this implies that there is no reasonable notion of Lebesgue measure on an infinite dimensional Hilbert space.

3. \( E \left[ \int_0^t W_s ds \right] = 0 \).

**Proof:** First, we prove a preliminary result:
Claim 2.1 Suppose $\alpha \geq 2$. Then,

$$W_t^\alpha = \alpha \int_0^t W_s^{\alpha-1} dW_s + \frac{1}{2} \alpha (\alpha - 1) \int_0^t W_s^{\alpha-2} ds.$$  

**Proof of Claim:** Letting $f(x) = x^\alpha$ we have by Ito’s formula:

$$dW_t^\alpha = df(W_t) = \alpha W_t^{\alpha-1} dW_t + \frac{1}{2} \alpha (\alpha - 1) W_t^{\alpha-2} dt$$  

$$\Rightarrow W_t^\alpha = \alpha \int_0^t W_s^{\alpha-1} dW_s + \frac{1}{2} \alpha (\alpha - 1) \int_0^t W_s^{\alpha-2} ds.$$  

By the claim, we have after letting $\alpha = 3$,

$$W_t^3 = 3 \int_0^t W_s^2 dW_s + 3 \int_0^t W_s ds$$  

$$\Rightarrow \int_0^t W_s ds = \frac{1}{3} \left( W_t^3 - 3 \int_0^t W_s^2 dW_s \right).$$  

Now, via the integration by parts formula, and the result $\int_0^t W_s dW_s = \frac{1}{2} (W_t^2 - t)$:

$$\int_0^t W_s^2 dW_s = \int_0^t W_s dW_s s - \int_0^t W_s^2 dW_s = \int_0^t W_s d \left[ \frac{1}{2} (W_s^2 - s) \right] = W_t \frac{1}{2} (W_t^2 - t) - \frac{1}{2} \int_0^t (W_s^2 - s) dW_s$$  

$$\Rightarrow \frac{3}{2} \int_0^t W_s^2 dW_s = \frac{W_t (W_t^2 - t)}{2} + \frac{1}{2} \int_0^t s dW_s$$  

$$\Rightarrow \int_0^t W_s^2 dW_s = \frac{1}{3} \left[ W_t (W_t^2 - t) + \int_0^t s dW_s \right]$$  

Now, $\int_0^t s dW_s$ is a Wiener integral, so that $W_t^3 = \int_0^t s dW_s \sim N \left( 0, \frac{t^3}{3} \right)$. Thus,

$$\int_0^t W_s^2 dW_s = \frac{1}{3} \left[ W_t (W_t^2 - t) + W_t^3 \right]$$  

so that

$$\int_0^t W_s ds = \frac{W_t^3 - \left[ W_t (W_t^2 - t) + W_t^3 \right]}{3}$$  

$$= \frac{tW_t - W_t^3}{3}$$  

Thus,

$$E \left[ \int_0^t W_s ds \right] = 0.$$  

□
3 Geometry

1. Let $\nabla$ be the Levi-Civita connection. Then, $\Gamma^j_{jk} = \partial_k \ln |g|^\frac{1}{2}$ where $g = \det g$.

Proof:
First, note that $g^{ij} = \frac{1}{g} \frac{\partial g}{\partial g_{ij}}$. To see this, note that by definition:

$$g^{ij} g_{jk} = \delta^i_k$$

implies:

$$g(g^{ij}) = e_j$$

where $(g^{ij})$ denotes the $j$-th column of $g^{-1}$ and $e_j$ is the column vector with entries zero everywhere except position $j$ which is 1.

By Cramer’s rule:

$$g^{ij} = \frac{\det g_i}{\det g}$$

where $g_i$ is the matrix obtained by replacing the $i$-th column of $g$ with $e_j$.

Now, recall the definition of determinant determined by:

$$\det A = \sum_\sigma \text{sgn} \sigma A(1, \sigma_1) \cdots A(n, \sigma_n).$$

Then,

$$\frac{\partial \det g}{\partial g_{ij}} = \sum_{\sigma, \sigma_1 = j} \text{sgn} \sigma g(1, \sigma_1) \cdots g(n, \sigma_n)$$

where in the sum, we omit the terms $g(i, \sigma_i)$.

On the other hand,

$$\det g_i = \det g_i^T = \sum_{\sigma, \sigma_i = j} \text{sgn} \sigma g_i^T(1, \sigma_1) \cdots g_i^T(n, \sigma_n)$$

where in the sum, we omit the terms $g_i^T(i, \sigma_i)$. With this convention, by symmetry of $g$ we have:

$$\sum_{\sigma, \sigma_i = j} \text{sgn} \sigma g_i^T(1, \sigma_1) \cdots g_i^T(n, \sigma_n) = \sum_{\sigma, \sigma_i = j} \text{sgn} \sigma g_i(1, \sigma_1) \cdots g_i(n, \sigma_n).$$

Thus, $\det g_i = \frac{\partial \det g}{\partial g_{ij}}$, so that

$$g^{ij} = \frac{1}{g} \frac{\partial g}{\partial g_{ij}}.$$

Now,

$$\partial_k g_{mj} = \partial_k \langle \partial_m, \partial_j \rangle = \langle \nabla_{\partial_k} \partial_m, \partial_j \rangle + \langle \partial_m, \nabla_{\partial_k} \partial_j \rangle = \langle \Gamma^l_{km} \partial_l, \partial_j \rangle + \langle \partial_m, \Gamma^l_{kj} \partial_l \rangle.$$
\[ \Rightarrow \partial_k g_{mj} = \Gamma^l_{km} g_{lj} + \Gamma^l_{kj} g_{ml}. \] (7)

Thus,

\[
g^{im} (\partial_k g_{mj}) = \Gamma^l_{km} g_{lj} g^{jm} + \Gamma^l_{kj} g_{ml} g^{jm} = \Gamma^l_{km} \delta^m_l + \Gamma^l_{kj} \delta^j_m = \Gamma^l_{kl} + \Gamma^l_{kl} = 2 \Gamma^l_{kl}
\]

\[ \Rightarrow \Gamma^l_{lk} = \frac{1}{2} g^{im} (\partial_k g_{mj}). \]

So:

\[ \Gamma^l_{lk} = \frac{1}{2} g^{im} (\partial_k g_{mj}) = \frac{1}{2g} \frac{\partial g}{\partial x^k} = \frac{\partial}{\partial x^k} \ln |g|^{\frac{1}{2}} \]

where in the third equality, we have used the chain rule. □

**Remark:** In particular: \( \frac{\partial g}{\partial x_k} = 2 \Gamma^l_{lk}. \)

2. \( \partial_i g^{il} = -g^{ij} \Gamma^l_{ij} - g^{lk} \Gamma^l_{ik} \) (metric \( g^{ij} \) with Levi-Civita connection).

**Proof:**

\[ \delta^i_k = g^{ij} g_{jk} \Rightarrow 0 = \partial_i \left( g^{ij} g_{jk} \right) = \partial_i g^{ij} g_{jk} + g^{ij} \partial_i g_{jk} \Rightarrow \partial_i g^{ij} g_{jk} = -g^{ij} \partial_i g_{jk} \]

\[ \Rightarrow \partial_i g^{il} = \delta^l_j \partial_i g^{lj} = g^{lk} g_{jk} \partial_i g^{lj} = -g^{ij} g^{lk} \partial_i g_{jk}. \]

From (5) of exercise 8:

\[ \partial_i g_{jk} = \Gamma^s_{ij} g_{sk} + \Gamma^s_{ik} g_{js} \]

\[ \Rightarrow \partial_i g^{il} = -g^{ij} g^{lk} \partial_i g_{jk} = -g^{ij} g^{lk} (\Gamma^s_{ij} g_{sk} + \Gamma^s_{ik} g_{js}) \]

\[ = -g^{ij} \Gamma^s_{ij} \delta^l_s - g^{lk} \Gamma^s_{ik} \delta^l_s = -g^{ij} \Gamma^l_{ij} - g^{lk} \Gamma^l_{ik}. \] □