3-2: The Integral Test

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1 Lab Questions

2 Motivation

- Let's consider the series $S_p = \sum_{j=1}^{\infty} \frac{1}{j^p}$ for any positive real number. Does it converge? How do we show it? If it does converge, then we can interpret $q_{p,j} = \frac{1}{S_p j^p}$ as the probability mass function of a random variable, and use it to answer any probabilistic questions.

- The idea is to compare the series to an integral, and then use what we know about integration and its relationship to Riemann sums to determine whether or not the series converges.

3 The Integral Test

- In lab yesterday, you learned how to integrate functions with an infinite upper bound. In this lecture, we’ll apply that theory to determine whether or not a series (for example, of the type in §1) converges.

- Suppose $\sum_{k=1}^{\infty} a_k$ is a series with positive terms; that is, $a_k > 0$. Suppose $f(x)$ is a decreasing, continuous function defined for $x \geq 1$ for which $f(k) = a_k$; i.e. the value of $f$ at the integer $k$ is just the $k^{th}$ term of the series.
If we consider the right-handed sums, or the sum of the areas in the boxes above, we know that
\[
\sum_{k=2}^{n} a_k = \sum_{k=2}^{n} f(k) \Delta x \leq \int_{1}^{n} f(x) dx \quad \text{for all } n
\]
\[
\Rightarrow \sum_{k=2}^{\infty} a_k \leq \int_{1}^{\infty} f(x) dx.
\]
(1)

So, if we know the integral on the RHS of (1) is finite, then we know the series on the LHS of (1) is bounded above, and therefore converges. Notice also this shows that if the series diverges, then the integral must diverge (for instance, let \(a_k = \frac{1}{k}\)).

Similarly, if we consider left-handed sums, as above, we can show that \(\int_{1}^{\infty} f(x) dx \leq \sum_{k=1}^{\infty} a_k\).

**Theorem 1** *(The Integral Test)* Suppose that \(\sum_{k=1}^{\infty} a_k\) is a series of positive terms, and that \(f(x)\) is a positive decreasing function for \(x \geq 1\). If \(f(k) = a_k\) for all \(k \geq 1\) then
\[
\sum_{k=2}^{\infty} a_k \leq \int_{1}^{\infty} f(x) dx \leq \sum_{k=1}^{\infty} a_k
\]
and therefore, either the series and the improper integral both diverge, or the series and the improper integral both converge.

§ Notice: The integral test only tells you convergence or divergence, not what the limit is (if it exists).

**\(\sum_{k=1}^{\infty} \frac{1}{k^p}\)**. Choose \(f(x) = \frac{1}{x^p}\), and note that \(f(x)\) is continuous and decreasing for \(x \geq 1\). Then:
\[
\int_{1}^{\infty} \frac{dx}{x^p} = \lim_{N \to \infty} \int_{1}^{N} x^{-p} dx = \lim_{N \to \infty} \left( \frac{1}{p-1} \right) \left( 1 - N^{-p+1} \right).
\]
If \(p > 1\) then \(-p + 1\) is negative so that \(N^{-p+1} \to 0\) as \(N \to \infty\). Thus, \(\int_{1}^{\infty} \frac{dx}{x^p} = \frac{1}{1-p}\) is finite, and the upper bound in the integral test implies that \(\sum_{k=1}^{\infty} \frac{1}{k^p}\) is converges for \(p > 1\).
If \(0 < p < 1\) then we can write
\[
\int_{1}^{\infty} \frac{dx}{x^p} = \lim_{N \to \infty} \left( \frac{1}{1-p} \right) (N^{-p+1} - 1).
\]
Since \(-p + 1 > 0\) this grows without bound as \(N \to \infty\), and so the integral is infinite. By the lower bound in the integral test, we’re forced to conclude that the series \(\sum_{k=1}^{\infty} \frac{1}{k^p}\) diverges for \(0 < p < 1\).

The case \(p = 1\) is the harmonic series which we’ve already covered; but we can use the integral test also to establish its divergence [why?].
• Recall that the convergence or divergence of a series is not determined by finitely many terms; only the tail of the series matters. Thus, it is sufficient to apply the integral test to the series one gets by omitting finitely many terms.

  - Consider \( \sum_{k=1}^{\infty} \frac{1}{k^2} \). The second term is larger than the first, so we can’t directly apply the integral test. But, the series is decreasing for \( k \geq 2 \) because the function \( f(x) = x^2 e^{-x} \) is decreasing for \( x > 2 \) (why?). So, by the integral test:

\[
\sum_{k=3}^{\infty} k^2 e^{-k} \leq \int_{2}^{\infty} x^2 e^{-x} \, dx
\]

a straightforward integral by parts of which yields that the series is convergent.

• (Error bounds) Suppose \( S = \sum_{k=1}^{\infty} a_k \) is convergent, and I want to approximate \( S \) by \( S_n = \sum_{k=1}^{n} a_k \). What is the error incurred by this approximation? That is, what is the error \( E_n = S - S_n \)?

  - Notice that the error \( E_n = \sum_{k=n+1}^{\infty} a_k \). If \( f \) is a continuous, decreasing function for \( x \geq n \) and \( f(k) = a_k \) for all \( k \geq n \) then

\[
\sum_{k=n+1}^{\infty} a_k \leq \int_{n}^{\infty} f(x) \, dx
\]

so that the integral gives an upper bound on the error \( E_n \). If we want a lower bound:

\[
\sum_{k=n+1}^{\infty} a_k \geq \int_{n+1}^{\infty} f(x) \, dx.
\]

Thus:

  - **Theorem 2 (Error Estimate for the Integral Test)**

\[
\int_{n+1}^{\infty} f(x) \, dx \leq E_n \leq \int_{n}^{\infty} f(x) \, dx.
\]

• Consider approximating \( \sum_{k=1}^{\infty} \frac{1}{k^2} \) by \( S_{10} = \sum_{k=1}^{10} \frac{1}{k^2} = 1.55 \). How close is our approximation?

  - Using the above estimate:

\[
\int_{11}^{\infty} \frac{dx}{x^2} \leq E_{10} = \sum_{k=11}^{\infty} \frac{1}{k^2} \leq \int_{10}^{\infty} \frac{dx}{x^2}.
\]

Evaluation of the integrals tells us that \( E_{10} \) is between \( \frac{1}{11} \) and \( \frac{1}{10} \). Thus, the sum of the series lies between \( 1.55 + \frac{1}{11} \) and \( 1.55 + \frac{1}{10} \).