Section 6.2: Inference for $\mu_1 - \mu_2$: Independence

- Examples:

1. Conduct a preliminary test using a 5% level of significance with $n_1 = 21$ and $n_2 = 20$ if $F = \frac{s_1^2}{s_2^2} = 1.00$.
   
   **Solution:** We need to determine if,
   
   $$\frac{1}{F_{1-\frac{\alpha}{2}}} < F = 1.00 < F_{1-\frac{\alpha}{2}}.$$

   In this case, $\alpha = 0.05$ so that,

   $$\mathbb{P}[F > F_{1-\frac{\alpha}{2}}] = 1 - \left(1 - \frac{\alpha}{2}\right) = \frac{\alpha}{2} = 0.025$$

   Looking this up in the table with $df_1 = n_1 - 1 = 21 - 1 = 20$ and $df_2 = n_2 - 1 = 20 - 1 = 19$ we find

   $$F_{1-\frac{\alpha}{2}} = 2.51, \quad \frac{1}{F_{1-\frac{\alpha}{2}}} = 0.398.$$ 

   Since

   $$0.398 < 1.00 < 2.51$$

   we’re forced to accept $H_0$.

2. Recall the curriculum in the law school example concerning men and women:

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Males</th>
<th>Females</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample Size</td>
<td>15</td>
<td>12</td>
</tr>
<tr>
<td>Mean</td>
<td>15.8</td>
<td>12.4</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>4.2</td>
<td>3.6</td>
</tr>
</tbody>
</table>

   Suppose that an evaluation committee is concerned that the new curriculum may be differentially effective among male and female students. Test this hypothesis with a 5% level of significance.
Solution: We’ll assume here that $\sigma_1 = \sigma_2$ but that they’re unknown since $s_1 \approx s_2$ and the sample size is small.

Our test statistic is,

$$t_0 = \frac{\bar{X}_1 - \bar{X}_2}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{15.8 - 12.4}{3.9473 \sqrt{\frac{1}{15} + \frac{1}{12}}} = 2.2240.$$

Since $d_f = n_1 + n_2 - 2 = 15 + 12 - 2 = 25$ and this is a two-sided test, $t_{2.5} = 2.06$. Since $t_0 > t_{2.5}$ we’re forced to reject $H_0$, meaning that there is evidence at the 5% level of significance that the test is different on males versus females.

3. (# 3) To investigate the numbers of hours that graduate students work in addition to full time class loads, a random sample of 20 male graduate students is selected who work a mean of 16.4 hours per week with variance 4.7 hours. A second random sample of 20 female graduate students is selected who work with a mean of 13.8 hours per week with a variance of 6.1 hours. Construct a 95% confidence interval for the difference in mean number of hours per week between male and female graduate students. Assume that the variances are equal.

Solution: The confidence interval takes the form,

$$(\bar{X}_1 - \bar{X}_2) \pm t_{1-\frac{\alpha}{2}} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$= (16.4 - 13.8) \pm 2.024 \cdot 2.3238 \sqrt{\frac{1}{20} + \frac{1}{20}} = 2.6 \pm 1.5$$

So, male graduate students work on average $1.1 - 4.1$ hours more than female graduate students with 95% confidence.

4. (# 7) A randomized trial is conducted to evaluate the effectiveness of a newly developed treatment for joint pain in patients with arthritis. The treatment is compared to an established treatment that has been shown to be effective. Two hundred patients with arthritis agree to participate and are randomly assigned to either a newly developed treatment or to the established treatment. Among the several clinical outcomes in the investigation, we focus on a secondary outcome: the quality of life (QOL). QOL scores ranging from 0 to 100 are measured on each patient (the larger the better). The outcomes are as follows:

<table>
<thead>
<tr>
<th>Treatment</th>
<th># of Patients</th>
<th>Mean QOL</th>
<th>SD QOL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Newly Developed</td>
<td>100</td>
<td>80.2</td>
<td>5.7</td>
</tr>
<tr>
<td>Established</td>
<td>100</td>
<td>75.4</td>
<td>6.1</td>
</tr>
</tbody>
</table>

(a) Construct a 95% confidence interval for the difference in mean QOL scores between treatments.

(b) Is there enough evidence to support that QOL scores are for the new treatment are better than the QOL scores for the established treatment at a 5% level significance?

Solution: We’ll assume the population variances are equal.

(a) Since $n_1, n_2 \geq 30$, the confidence interval is of the form,

$$(X_1 - X_2) \pm Z_{1-\frac{\alpha}{2}} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$= (80.2 - 75.4) \pm 1.96 \cdot 5.9034 \sqrt{\frac{1}{100} + \frac{1}{100}} = 4.8 \pm 1.6.$$
Thus, the mean QOL score in the newly developed group is between 3.2 to 6.4 points higher than the established group with 95% confidence.

(b) 

\[ H_0 : \mu_1 = \mu_2 \]
\[ H_1 : \mu_1 > \mu_2 \]

so that this test is a right tailed test.

The test statistic is,

\[ t_0 = \frac{\bar{X}_1 - \bar{X}_2}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{80.2 - 75.4}{5.9034 \sqrt{\frac{1}{100} + \frac{1}{100}}} = 5.7492 \]

\( df = 198 \), so that \( t_\alpha = 1.645 \). Since \( t_0 > t_\alpha \) and this is a right tailed test, we reject \( H_0 \) and conclude that there is evidence at the 5% level of significance to conclude that the mean QOL for the newly developed treatment is higher than the mean QOL for the established treatment.

5. An engineer wants to know whether the strength of two different concrete mix designs differ significantly. He randomly selects 9 cylinders measuring 6 inches in diameter and 12 inches in height into which mixture 1 is poured into, and 10 cylinders of which mixture 2 is poured into. The results are as follows:

<table>
<thead>
<tr>
<th>Mixture 1</th>
<th>Mixture 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>3960</td>
<td>4090</td>
</tr>
<tr>
<td>3100</td>
<td>4070</td>
</tr>
<tr>
<td>3830</td>
<td>4890</td>
</tr>
<tr>
<td>4080</td>
<td>5020</td>
</tr>
<tr>
<td>4330</td>
<td>4120</td>
</tr>
<tr>
<td>3780</td>
<td>4620</td>
</tr>
<tr>
<td>2940</td>
<td>4190</td>
</tr>
</tbody>
</table>

(a) Construct a 95% confidence interval.

(b) Is there enough evidence to support the hypothesis that the two mixtures have the same strength at the 2% level of significance?

Solution: Both populations are approximately normal and free of outliers.

![Figure 1: Normal probability plot and box and whisker plot for cylinder 1.](image)

First, we need to conduct the preliminary test at the 5% confidence level.
Figure 2: Normal probability plot and box and whisker plot for cylinder 2.

Cylinder 1: $\bar{X}_1 = 3669, s_1 = 458$.
Cylinder 2: $\bar{X}_2 = 4483, s_2 = 474$.

$F = \frac{s_1^2}{s_2^2} = 0.9336$. At a 5% level of significance and $df_1 = 8, df_2 = 9$,

$$F_{1-\frac{\alpha}{2}} = 4.10, \quad \frac{1}{F_{1-\frac{\alpha}{2}}} = 0.2439$$

Since

$$0.2439 < 0.9336 < 4.10$$

we accept $H_0$ and conclude that $\sigma_1 = \sigma_2$ at the 5% level of significance.

(a) The confidence interval in this case is,

$$(\bar{X}_1 - \bar{X}_2) \pm t_{1-\frac{\alpha}{2}} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$= (3669 - 4483) \pm 2.11 \cdot 466.5959 \cdot \sqrt{\frac{1}{9} + \frac{1}{10}} = -814 \pm 452$$

Thus, cylinder 2 is stronger by 362 to 1266 stronger than cylinder 1 with 95% confidence.

(b) This is a two-tailed test. The test statistic is,

$$t_0 = \frac{\bar{X}_1 - \bar{X}_2}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{3669 - 4483}{214.3860 \sqrt{\frac{1}{9} + \frac{1}{10}}} = -8.2637.$$ 

df = 17 so that $t_{2\%} = 2.567$. Since $t_0 < -t_{2\%}$ we reject the null hypothesis and conclude that there is sufficient evidence at the 2% level of significance that the two cylinders do not have the same strength.