1 Introduction

You already know a lot of mathematics. You have been receiving formal training in mathematics since your first day of school and you have already mastered more mathematical skills than most of the humans who have ever lived. You already have extremely powerful tools. However, there are two questions that you have probably not answered (or maybe even been asked) in all your mathematics classes:

1. What is mathematics and how is it different from other subjects?

2. Why is mathematics important, (or, to me the same question, how can I use the mathematics I already know to do interesting things)?

The goal of this course is to begin to answer these questions.

For us “mathematics” means the “mathematical sciences”, which includes traditional mathematics along with statistics and computer science. These three areas share common roots and many common tools and techniques. Our examples will come from all three of these areas.

To paraphrase *Shrek*, mathematics is like an onion—mathematics does not smell bad when you leave it in the sun or make you cry—mathematics has layers. The process of doing mathematics is one of peeling back those layers, giving a more and more detailed understanding of some aspect of our world.

All good mathematics starts with a problem. The more interesting the problem, the more important the mathematics. Of course “interesting” is a relative term. What is interesting to me may not be interesting to you and vice versa. However, everybody has questions that are interesting to themselves and probably many others. The problem can be esoteric or practical, but the first step should always be choosing the problem.

The first step in attacking the problem is the key step and it is probably not something you have done much of in your study of mathematics. Almost every problem, and certainly every problem that talks about the “real world”, is incredibly complicated. For example, many on-line book and movie sellers have built in features that recommend titles that may interest you. If you think about this problem for a minute or two, it is easy to come up with hundreds of factors that might influence preferences. For example, the season (are you looking for a beach read or a serious text), the economy (you may want a book on luxury yachts or a history of the Great Depression), the time of day of your search (if you are searching at 12 PM on Saturday then you might prefer a book on social networking), and so on. Including all these factors is hopeless.

So the first step in attacking the problem must be to “make a model” or create an abstraction of your problem. By this we mean that you must decide what influences are the most important and precisely describe what situations you will consider. It is this model problem that you actually study. A good model walks the line between being simple and
concrete enough to be understandable, but realistic enough to say interesting things about
the original problem. Much of the “art” of quantitative reasoning goes into the construction
of models and it is a process that is often repeated many times for the same problem. For
example, to study how a car might skid on a slippery road, you might start by considering
the motion of a unicycle, then a bicycle and finally a car. We build up more accurate and
more complicated models in layers.

Once the model has been precisely defined, then you can use the multitude of tools you
already have to study the model. In mathematics this process starts by looking at examples,
which is much like doing experiments in science. Next we make “conjectures”, statements
that we think are true about the model and which we observe consistently in the examples.
This is like creating “theories” in science. Next, and this is an aspect that is unique to
mathematics, if we can, we “prove” the conjectures. Proofs are portrayed as the bogeyman
of mathematics, but we will see that they are actually our saviors and ports in the storm.

Finally, we return to the original problem to see what we have learned. If the model
is precise enough and our results complete enough, we may move on to the next problem.
If not, then we add another layer, using what we have learned to make the model a more
accurate characterization of the original problem.

Keep this process in mind as we progress through the course. The process of model
building and creating layers of abstraction is tremendously powerful because it allows us to
attack extremely difficult problems by a sequence of steps that are not so difficult.
2 A Toy Problem

Recall the outline of how mathematics is done from the previous section

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2.1 Stating the Problem

We start with a “toy problem”– that is, a problem that at least doesn’t sound too hard and, on the face of it, looks like it is just for fun. One of the most wonderful things about mathematics is that some problems that look like toys turn out to have amazing applications. This is also one of the difficulties of mathematics–you can’t forget anything, even the toys, because the thing you forget will be just what you need some unpredictable time in the future.

In the 1700’s the town of Konigsberg, Prussia, (now Kaliningrad, Russia) had seven bridges that connected the banks and two islands in the Pregel River that made up the town. Legend has it that on sunny summer afternoons, the locals would amuse themselves by trying to take a walking tour of the town that crossed each bridge exactly once, ending where it started. This turns out to be harder than it looks–try it on the Fig. 1 below.

INSERT FIGURE 1 HERE

We can generalize this problem easily: Given any collection of rivers, islands and bridges, is there a walking path that crosses all the bridges exactly once and returns to the starting
point. We can imagine many different possible layouts with lots of bridges and lots of islands. Solving these problems one at a time might make an amusing puzzle for the newspaper, but it would be much nicer if we could give a method for solving them all at once.

An even more fundamental question is “Is there a solution?” If you tried to find a path around Konigsberg, you know this isn’t obvious. Is there a way to tell before trying all possible paths, if a path that crosses every bridge exactly once even exists?

If we are going to tackle many of these problems, including the more “abstract” sounding ones, then it pays to think about the general set up more carefully.

2.2 Building a Model

Now that we have our problem we can start the second step, building a model. That is, think about what the “real question” is—what is the essence of the problem, what features are essential and what are just fluff.

Looking at the Konigsberg bridge problem, we first notice that what is important are the bridges. How we walk around on land, either the banks of the river or the island(s), doesn’t matter. It is only crossing a bridge that we care about.

So we can start our abstraction by reducing the island to just one small dot (see Fig. 2). We can also think of the bridges that end on the banks of the river as ending at the same point if they reach the same land mass—that is, push the ends of the bridges together that are connected by a walk that is entirely on land (see Figs 3). We can even get rid of the river if we agree that we will only walk on the bridges, changing from one bridge to the next at the dots at the ends of the bridges (see Fig. 4).

INSERT FIGURE 2 HERE

INSERT FIGURE 3 HERE

What we are left with is a much simpler, more abstract, picture, but it catches all the essential features of the more complicated map. In order to talk about pictures like Fig. 4, we should come up with some vocabulary—this is almost biblical. We feel like we understand things better if we have given them names. Luckily there already is a standard terminology
for pictures like this...unluckily it uses words that are already used in mathematics, so we have to get used to the new meanings.

The reused word is “Graph”. You’ve used this word to describe pictures of functions and we will do some graphing of functions next week. Here, we use the word “graph” to describe a different idea. This is the standard usage.

**Definition:** A *graph* is a collection of collection of points, called *nodes* or *vertices* and curves that connect pairs of vertices to each other called *edges*.

The simplest way to specify a graph is to give names to the nodes and edges, then specify the edges by saying which two nodes each edge connects. If we label the nodes in Fig. 4 as $A, B, C$ and $D$ and edges as $a, b, c, d, e, f$ and $g$ as shown in Fig. 5, then

- edges $a$ and $b$ connect node $A$ to node $B$,
- edges $c$ and $d$ connect node $B$ to node $C$,
- edge $e$ connects node $C$ to node $D$,
- edge $f$ connects node $B$ to node $D$, and
- edge $g$ connects node $D$ to node $A$.

**Notes:** 1. The edges are all “two way streets” in this definition. So an edge that connects $A$ to $B$ also connects $B$ to $A$. 

INSERT FIGURE 5 HERE

The information given above, the list of nodes, the list of edges and the description of which nodes each edge connect, gives all the important information about the graph, at least for our walking tour problem. So for the purposes of this problem, the two graphs pictured in Fig. 6 are “the same”. Even though they look different, the list of nodes and edges and which nodes are connected are exactly the same. Hence, we can solve our problem on one graph if and only if we can solve it on the other. INSERT FIGURE 6 HERE
2. When we draw a graph, the nodes are precisely where we say they are—just because two edges cross it does not imply that the crossing is a node. So Fig. 6 shows two identical graphs.

2.3 Using this Model

Now we can take any village with its collection of islands and bridges and create a graph that represents how the land masses are connected by bridges. We can even interpret questions about the town and bridges in terms of the corresponding graph.

What kinds of questions can we ask about graphs? We will see later in the course that graphs can be used to describe relationships between lots of different things. For example, you can think of all the people on Facebook as the nodes and the edges as the “friend” relationship. You can think of all the different species as nodes and edges represent “node A eats node B”, and so forth. Right now we will stick with the map/bridge interpretation, but it is interesting to think about what the concepts below mean for other applications.

One natural question about any graph is “can we get from one node to another along edges?”. To help us talk about this idea we give the definition of a “path” as follows:

Definition: A path on a graph is a list of node, edge, node, edge, node etc. such that each pair of nodes is connected by the edge that separates them.

So on the graph if Fig. 5 is $A, a, B, d, C$ is a path (go from $A$ to $B$ via edge $a$, $B$ to $C$ via edge $D$), but $A, a, B, b, C$ is not a path because edge $b$ does not connect node $B$ to node $C$.

Graphs which don’t have paths from every node to every other node are common in maps (“you can’t get there from here”...e.g., you can’t drive from Boston to London). Such a graph is called disconnected. On the other hand, if you can get from any node of a graph to any other node of the graph by some path, then the graph is called connected. (Contemplate what would mean to say the Facebook graph is “connected” or “disconnected”.)

Let’s start our study of graphs by thinking about paths on a connected graph. If I want to go from some node called $A$ to some node called $B$ then the fact that the graph is connected means that there is a path. But just because there is a path, it does not mean it is a short or convenient path.

For example, in the graph in Fig. 5, the path $AaBabCeD$ connects the node $A$ to $D$. However, this is certainly not the most efficient path, since the path $AgD$ also connects $A$ and $D$. The second path is more efficient because we do not repeat any edges or nodes. As a first question about paths on graphs, we can ask

On a connected graph, is every pair of different nodes connected by a path that does not repeat any edges or nodes?

This is like asking if there is an “efficient” path between nodes on a connected graph or are we doomed to use some edges or nodes many times?
If we look at some examples, then it seems automatic that the answer is “yes, sure”. For example, the graph in Fig. 7, there are several paths from A to E which don’t repeat any edges or nodes.

If this were science, we would be happy at this point. We have an object of study (“graphs”), we have a theory (“For each pair of nodes on a connected graph, there a path from one to the other that doesn’t repeat nodes or edges”), and, finally, we have evidence for our theory–examples above (“experimental evidence”)... We would look at more examples and become more and more confident of our theory because the examples always follow the theory.

But are we really sure? Is there some doubt left? If we were to draw a connected graph with a million nodes and ten trillion edges, would we always be able to a path without repeating nodes or edges between any of the pairs of nodes?

This is where math differs from science and from all other subjects for that matter. The process above of model building and studying examples is familiar in many fields–but in math, to really accept a theory as true we need a “proof”.

A proof is nothing scary–a proof is just an explanation of why a certain statement MUST be true. Once we have a proof, then we know for sure that the statement is true and will always be true for any example anywhere, no question, bet your life and this is why mathematics is so successful and so permanent. The ancients were just as smart as we are, but we chuckle at their chemistry (earth-air-fire-water), we marvel at their ability to compute positions of the planets even though they thought the sun went around the earth, but we all learned their geometry in high school– it is just as true now as it was 2000+ years ago. And because that geometry was so solidly built on proof, we have been able to use it to build more advanced mathematics that enables our world.

So, lets give a proof:

2.4 Comments on Proof

We use the word proof in common language very sloppily, so we must learn to identify what is and what isn’t a proof in mathematics.

A proof is a sequence of statements, each one is either part of our model (in geometry you called them axioms), or it is a statement we have already proven, or it follows by the usual rules of logic from the previous statements. The classic first example of a proof is
All men are mortal.
Socrates is a man.
Socrates is mortal.

The first two statements here are our model of the world. People, by definition, are mortal is part of our model. We are assuming that Socrates is a man, not an alien. Therefore, we are stuck with concluding that Socrates is mortal. This is a good example of a direct proof where a general statement (all men are mortal) is used to prove a particular statement (Socrates is mortal).

The most common error when giving “proofs” in everyday language is using an example (or examples) to try to prove a general statement. (E.g., “Socrates must be mortal because everyone I know is mortal.”) We accept this kind of argument, for good reason (e.g., the last three people that walked down that path at night were eaten by tigers, so I’m not going to walk down that path at night because if I do I will be eaten by a tiger.) However, we have to recognize that examples only prove the existence of examples. No matter how many examples we look at, they never, ever prove a general statement. To prove a general statement, we must know why the statement must be true, not just that there are examples.

You will probably find this the most difficult habit to overcome. While we can learn a lot by generalizing from examples (and that is what scientific experimentation is all about), examples never provide a proof. Always make sure that when giving a proof of a general statement, you don’t try to generalize from specific examples, rather, make sure that your argument applies to all possible examples.

2.5 Proof

One of the difficulties of teaching about proofs is that it makes sense to start out with simple proofs...the problem is that if the proof is simple then there is a real question of why there needs to be a proof in the first place (and then mathematicians look like they are just making a simple thing difficult). We’ll try to avoid this, but if you find yourself saying “Oh, I already believe this”, make sure you ask yourself “but why do I believe it?”. If you can’t give the reason in a clear set of sentences that anyone could understand, that your statement must be true then you have faith but not knowledge.

As a first example, we prove the following:

For a connected graph, any pair of nodes are connected by a path that does not repeat any nodes or edges.

When we read the problem carefully, we see that it is a very general statement. It says that for any pair of nodes in any connected graph there is a path in the graph that connects the nodes without repeating nodes or edges. We can’t possibly prove this statement by giving examples since there is no way to cover all the possible examples! We must find a way
to describe why this statement must be true for every connected graph without referring to a particular graph.

This isn’t so easy. In some ways you can think of this as a writing exercise. Don’t fall into old habits, but think carefully about keeping everything general.

Start with any connected graph and any pair of nodes. We can refer to the nodes as node $A$ and node $B$. This doesn’t effect the generality of the argument since if the names were different, we could just change them to $A$ and $B$.

Since the graph is connected, we know (by the definition of connected graph) that there is some path that starts at $A$ and ends at $B$. However, this path might involve repeats of many edges and nodes. Let $C$ be one of the repeated nodes (again, changing the name of a node if necessary). Then the path must look like

$$A, \text{edge}, \text{node}, \ldots, \text{edge}, C, \text{edge}, \text{node}, \ldots, \text{edge}, C, \text{edge}, \text{node}, \ldots, \text{edge}, B$$

that is, it starts with $A$, has edges and nodes alternating until $C$, then more edges and nodes until $C$ occurs for a last time, then edges and nodes until $B$. There could be several occurrences of node $C$ between the first and last (Fig. 8).

Since the node $C$ is visited more than once in the path, the part of the path that starts and ends at $C$ forms a loop. We can make a shorter path from $A$ to $B$ by cutting out this part of the path. That is, form a new path from $A$ to $B$ which starts out the same as our original path, but we remove all the nodes and edges between the first and last occurrences of the node $C$.

This is definitely a shorter path from $A$ to $B$. It might or might not have repeated nodes. If it does not have any repeated nodes then it can’t have any repeated edges either since if an edge is repeated, the nodes at its ends must also be repeated. Hence we are done—we have found a path from $A$ to $B$ with no repeated edges or nodes.

If, on the other hand, there are still repeated nodes, we can remove them just as we did above, cutting out another unneeded loop along the way from $A$ to $B$. Again we make the path shorter. This process of cutting out loops must eventually end because our path can’t keep getting smaller forever. Hence, this process must leave a path from $A$ to $B$ with no repeated nodes or edges.

This completes the proof. We started with any pair of nodes in any connected graph and have shown there must be a path between our two chosen nodes that does not have any repeated edges or nodes.
We now know WHY our original statement is true—we don’t need to look at examples and test it with experiments, we are sure. This proof is particularly nice because it also gives us a way to start with a path that has repeats and cut it down to a path without repeats (such proofs are sometimes called “constructive”).

Once we realize why the statement is always true, everything is clear and seems really easy. That’s the way it should be—things we understand always look easy, but it is the process of explaining carefully why and how they work that makes them easy.

2.6 Restating the Konigsberg Problem

Now let’s go back to the Konigsberg bridge problem, which is considerably harder. We can now state the general version of this problem.

First problem: Is there a path on the Konigsberg bridge graph which starts and ends at the same node and which uses every edge exactly once?

Note that this is a very specific problem about a very specific graph. We can make the problem more general, and more interesting, as follows:

Second problem: Given a graph, can you tell if there is a path that starts and ends at the same node and uses every edge exactly once?

This is an interesting way to state the problem since we are not asking what the path is, just if there exists one. It is conceivable we can answer “yes, there is such a path” without knowing what the path is. (One can think of all sorts of implications of the existence of a proof of this sort of statement.)

Also, is it possible to prove the answer is “No”, that is, “No such path exists.”? Can we “prove a negative”? In order to prove the negative, do we have to try every possible path?

While these questions can be debated by themselves, we will stick with the concrete. But think back on these questions when we figure out the problems above.

2.7 Working on the Problem

These problems, motivated by Koinigsberg, were first asked, and answered in the 1700’s by Leonhard Euler, one of the many dead, white, male, European mathematicians of 1600-1800’s. Most of the mathematics you have learned is due to work of mathematicians for which most of these adjectives apply (and if not all of them apply, then they have probably been dead for much longer). One of my favorite things about this course is that some (in fact, a fairly large proportion) of the mathematics we discuss will be work of mathematicians for which a number of these adjectives do not apply. Mathematics is still alive, well and growing quite nicely, thank you.

Terminology: In a connected graph, we call a path that begins and ends at the same node and crosses every edge exactly once an Euler circuit.
We start by looking at some examples. We can't give any sort of general proof with examples, of course(!), but we can hopefully get some ideas about how things work.

The people of Konigsberg tried hard to find an Euler circuit and no one ever did. I've tried and always seem to end up leaving out at least one bridge, but being "stuck" (to leave the node I'm in I have to use an edge I've already used). Try it.

It might be possible to make a list of every possible path from every possible starting point and show that every single one of them eventually ends up either leaving out a bridge or forcing one or more bridges to be crossed more than once.

But suppose a some bridges down? Say we remove bridges $f, b$ and $d$ (see Fig. 9). Now the path $AaBcCeDgA$ works great!

\[\text{INSERT FIGURE 9 HERE}\]

If instead of removing a bridge, we add some ones, say edge $h, i, j$ (see Fig 10). Now there is a path from $A$ to $A$ that doesn't repeat any edges. (Find it.)

\[\text{INSERT FIGURE 10 HERE}\]

So the trouble with the original Konigsberg graph isn't that it doesn't have enough edges or too many edges, the problem is that we get stuck. We are forced to visit nodes that we can't leave without crossing edges we've already used.

On the other hand, if we can always get out of a node that we enter, then we should be able to keep going until we have crossed every bridge.

Here we say "Ah Ha", maybe we have an idea. If a path exists that uses every edge exactly once, then every node that you "enter" via an edge, you must be able to "exit" by a different edge that you haven't used yet. So at each node, we can match up the edges—those that you arrive by and those that you leave by. So you can match up the edges in pairs. So the total number of edges touching the node must be an even number!

This is even true for the starting node. At the starting node, the first thing you do is "leave" via an edge. Every time you visit the node you both "enter" and "leave" by different edges. At the end you "enter" via the last edge—so again every "entering" edge can be paired with a "leaving" edge and the total number of edges touching the node must be an even number!
To make this precise, we first create a little more vocabulary:

The \textit{degree} if a node is the number of edges that have that node as an end point.

As often happens, as we think more carefully about a problem additional problems arise. For example, what if an edge begins and ends at the same node (both endpoints are the same?) We can decide, if we like, to forbid edges like this, or we can choose to allow them—it is our choice since we are making the model. We’ll allow them because they don’t cause any trouble for the problems we are dealing with now.

We can now take our ideas from above and make a precise “conjecture”—that is, a statement we think might be true. It is supported by examples and seems reasonable (a “theory” in science).

\textbf{Conjecture:} If a connected graph has one or more nodes with odd degree then there is no path that uses every edge exactly once and ends where it starts, i.e., it does not have an Euler Circuit.

Our proof will be carefully restating the argument above.

Suppose we have a connected graph with an Euler circuit and that \(A\) is any node in the graph. We need to show that the degree of \(A\) is even. There are two possibilities. Either \(A\) is the starting node (and hence the ending node) of the path, or it is one of the intermediate nodes.

Let’s think about the case where \(A\) is not the start/end node first. The path around the graph that uses every edge exactly once must visit \(A\), since there are edges that have \(A\) as an end point. In fact, this path could visit \(A\) several times. But every time the path arrives at \(A\), it must also leave \(A\) and since we can use each edge only once, it must leave by an edge that has not been used yet. That means that for every time \(A\) appears on the path, we “use” two edges, one to get into \(A\) and one to get out again. So every edge that touches \(A\) can be paired with a partner (the “entering” edge with the “exiting” edge along the path or “arriving edge” with the “leaving edge”). But since we can match the edges touching \(A\) up in pairs, the number of edges touching \(A\) must be even.

Now let’s deal with the other case. Suppose \(A\) is the starting (and ending) node. Well, we still have the same problem. We can match the first edge out of \(A\) on the path with the last edge returning to \(A\) at the end of the path and every other time \(A\) is visited, we can match the entering and exiting edges as above. So again, we can pair up the edges touching node \(A\) and the degree of \(A\) must be even!

So, if there is an Euler circuit in a graph then we see that every node must have even degree. We have given our proof without referring to any specific graph and the “node \(A\)” is completely general. The argument above applies to every graph with an Euler circuit and every node in that graph.

This completes the proof.

As an example, we see that the people of Konigsberg were doomed from the start. Every
node of the corresponding graph has odd degree, so there is no way for there to exist a path that crosses every bridge exactly once.

We have “proven the negative” statement—no matter how we try, if a connected graph has three or more nodes with odd degree, it can not have a path that uses every edge exactly once. So we do not have to check every possible path—we know there is no Euler circuit in Konigsberg.

2.8 Be Honest in Your Relationships

Like ever important relationship, you must always be honest. This is particularly important in mathematics since we are claiming to give permanent, cast iron proofs. The above statement is that if a graph has one or more nodes of odd degree then it can NOT have Euler circuit. The proof says nothing about the situation when all nodes have even degree. We could conjecture that such connected graphs have Euler paths, but we would have to prove this!

2.9 A related problem

Suppose we want to keep the citizens of Konigsberg active, so we propose a slightly different problem. We ask if there is a path around Konigsberg which uses every edge exactly ones, but we drop the requirement of ending where you start. Such a path is called an “Euler path” (to distinguish it from an Euler circuit).

In discussion section you (hopefully) came up with the following conjecture: If a connected graph has more than two nodes with odd degree then it does NOT have an Euler path.

In order to prove this conjecture we work by “contradiction”. Suppose we have a graph which has 3 or more nodes of odd degree and it does have an Euler path. Since the graph has nodes of odd degree then we know it does not have an Euler circuit, so the Euler path must end at a different node from where it starts. Let the starting node be called \( A \) and the ending node be called \( B \). Since the starting node \( A \) has an exiting edge that is not matched by an entering edge, it must have odd degree. Likewise, the ending node \( B \) must have an entering edge that is not matched by an exiting edge, so it too must have odd degree.

Now for the really clever part. Take the graph above and add one more edge from \( A \) to \( B \) (it is fine if there is already an edge connecting \( A \) and \( B \), just add another one). Then the degrees of \( A \) and \( B \) are both even, AND, the Euler path that starts at \( A \) and ends at \( B \) can be extended to an Euler circuit adding this new edge from \( B \) to \( A \) to the end of the Euler path (so the Euler circuit starts and ends at \( A \)).

But wait, our original graph had 3 or more nodes with odd degree. By adding an edge we made two of these nodes have even degree, but we also created an Euler circuit—so ALL nodes must have even degree. This is a contradiction because 3 minus 2 is not zero! There
is still a node of odd degree and we know such a node can not exist in a graph with an Euler circuit.

Hence, our original assumption that we had a connected graph with 3 or more nodes of odd degree that had an Euler path must be wrong! This is a “proof by contradiction”. If we assume the opposite of what we conjecture, we get a contradiction, so the conjecture must be true.

Note that again, we have NOT shown that a graph with exactly two nodes of odd degree has an Euler path, although this is a reasonable conjecture.