1. Prove that if $G$ is a simple group such that $61 \leq |G| \leq 70$, then $G$ is a cyclic group.
   [Feel free to use the Sylow theorems.]
   Referencing the lemmas from homework 9, we can create the following table:

<table>
<thead>
<tr>
<th>Order</th>
<th>Simple/Not Simple</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>61</td>
<td>Simple</td>
<td>Lemma P</td>
</tr>
<tr>
<td>62</td>
<td>Not Simple</td>
<td>Sylow’s Third Theorem: $n_7</td>
</tr>
<tr>
<td>63</td>
<td>Not Simple</td>
<td>Sylow’s Third Theorem: $n_7</td>
</tr>
<tr>
<td>64</td>
<td>Not Simple</td>
<td>Lemma Z: $64 = 2^6$</td>
</tr>
<tr>
<td>65</td>
<td>Not Simple</td>
<td>Lemma PQ: $65 = 5 \times 13$</td>
</tr>
<tr>
<td>66</td>
<td>Not Simple</td>
<td>Sylow’s Third Theorem: $n_{11}</td>
</tr>
<tr>
<td>67</td>
<td>Simple</td>
<td>Lemma P</td>
</tr>
<tr>
<td>68</td>
<td>Not Simple</td>
<td>Sylow’s Third Theorem: $n_{17}</td>
</tr>
<tr>
<td>69</td>
<td>Not Simple</td>
<td>Lemma PQ: $69 = 3 \times 23$</td>
</tr>
<tr>
<td>70</td>
<td>Not Simple</td>
<td>Sylow’s Third Theorem: $n_7</td>
</tr>
</tbody>
</table>

Notice that the only simple groups are those that have prime order. Since every group of prime order is cyclic, the proof is complete.

2. Let $A$ and $B$ be normal subgroups of a group $G$ such that $A \cap B = \{e\}$. Prove that $ab = ba$ for all $a \in A$ and $b \in B$.
   [Hint: Prove that $aba^{-1}b^{-1} \in A \cap B$.]

   - Note first that for all $a \in A$ and $b \in B$, we have $aba^{-1} \in B$ and $ba^{-1}b^{-1} \in A$. Correspondingly, $(aba^{-1})b^{-1} \in Bb^{-1} = B$ and $a(ba^{-1}b^{-1}) \in aA = A$. Thus, $aba^{-1}b^{-1} \in A \cap B$. Since $A \cap B = \{e\}$, we know that $aba^{-1}b^{-1} = e$, and therefore that $ab = ba$. This completes the proof.

3. Let $G$ be a group of size 15. Let $a$ be an element of order 3 and $b$ be an element of order 5. Set $A = \langle a \rangle$ and $B = \langle b \rangle$.

   (a) Prove that $AB = G$.

   - Since $A$ and $B$ have prime orders 3 and 5, each non-identity element of $A$ and $B$ also has an order of 3 and 5, respectively. Therefore $A \cap B = \{e\}$, for otherwise we could conclude $|A| = 5$ or $|B| = 3$, both of which would be contradictions.

   One way to accomplish our task is to determine if the set $AB = \{ab \mid a \in A$ and $b \in B\}$ has size 15. We can do this using properties of cosets. Let $A = \{e, a_1, a_2\}$. Then $B = eB \neq a_1B$ and $B = eB \neq a_2B$, since $a_1, a_2 \notin B$. Since different cosets are disjoint, the only case in
4. Let $G$ be a group of size $pq$ with $p$ and $q$ distinct primes with $p < q$ and $q \neq 1 \pmod{p}$. Prove that $G$ is cyclic.

[Hint: Mimic the arguments of the last question.]

Let $a$ be an element of order $p$ and let $b$ be an element of order $q$. Let $A = \langle a \rangle$ and $B = \langle b \rangle$.

• Lemma: $AB = G$.

(a) Since $A$ and $B$ have prime orders $p$ and $q$, they are cyclic and all non-identity elements generate their respective subgroup. Furthermore, since $p \neq q$, $A \cap B = \{e\}$, for otherwise we could conclude $|A| = q$ or $|B| = p$, both of which would be contradictions.
5. Let \( G \) have size \( 12 = 2^2 \cdot 3 \).

(a) If \( n_p \) equals the number of \( p \)-Sylow subgroups in \( G \), prove that \( n_2 = 1 \) or \( 3 \) and \( n_3 = 1 \) or \( 4 \).

- We use Sylow’s Third Theorem:
  - Since \( n_2 | 12 \), \( n_2 = 1, 2, 3, 4, 6, \) or \( 12 \). Of these, only \( 1 \) and \( 3 \) are equivalent to \( 1 \) (mod 2).
  - Since \( n_3 | 12 \), \( n_3 = 1, 2, 3, 4, 6, \) or \( 12 \). Of these, only \( 1 \) and \( 4 \) are equivalent to \( 1 \) (mod 3).

(b) Let \( A \) be a 2-Sylow subgroup and let \( B \) be a 3-Sylow subgroup. Prove that either \( A \) or \( B \) is normal.

[Hint: Otherwise \( n_2 = 3 \) and \( n_3 = 4 \). Show that \( G \) isn’t big enough to have so many \( p \)-Sylows.]

- Suppose that neither \( A \) nor \( B \) is normal. Then \( n_2 \neq 1 \) and \( n_3 \neq 1 \), so \( n_2 = 3 \) and \( n_3 = 4 \) by part (a). That \( n_3 = 4 \) means there are \( 4 \) distinct 3-Sylow subgroups. The two non-identity elements in any 3-Sylow subgroup generate that subgroup. Thus, any two distinct 3-Sylow subgroups can overlap only in the identity, for otherwise they would be the same subgroup. We conclude that the four distinct 3-Sylow subgroups account for \( 4 \cdot 2 + 1 = 9 \) distinct elements of \( G \). Now, choose any 2-Sylow subgroup. Since this subgroup has order 4, the three non-identity elements of \( A \) have orders 2 or 4, and, adding these to the identity and the eight elements of order 3, these bring the total number of elements of \( G \) to 12. Since the three non-identity elements of our 2-Sylow are the only elements of \( G \) with order 2 or 4, any other subgroup of order 4 must contain them, so that there is at most one subgroup of order 4. This contradicts that \( n_2 = 3 \), completing the proof.

(c) Prove that \( A \) and \( B \) are abelian. List all possible groups for \( A \) and \( B \).
First we consider $A$. Since $|A| = 4$ and 2 is a prime that divides 4, $A$ is guaranteed a subgroup of order 2, call it $H$. Since $2 \times 2 = 4$, $H$ has index 2 and is therefore normal (see HW 8, question 9.7), and is contained in $Z(A)$ (see HW 8, question 9.66). Since $Z(A)$ is a subgroup of $A$ and has at least 2 elements, either $|Z(A)| = 2$ or $|Z(A)| = 4$. If $|Z(A)| = 4$ then $Z(A) = A$ and we’re done. If $|Z(A)| = 2$ then $Z(A) = H$; form the factor group $A/H$ which has order 2 and is therefore cyclic. By the $G/Z$ Theorem, $A$ is abelian.

For $B$, note simply that $|B| = 3$, which is prime, so $B$ is cyclic and therefore abelian.

(d) If $n_2 = n_3 = 1$, prove that $G$ is abelian and thus $G = Z_3 \times Z_4$ or $G = Z_3 \times Z_2 \times Z_2$.

• Since $n_2 = n_3 = 1$, the 2-Sylow subgroup, call it $A$, and the 3-Sylow subgroup, call it $B$, are both normal. Furthermore, $A \cap B = \{e\}$, since non-identity elements of $A$ have order 2 or 4, whereas those of $B$ have order 3. Using the exact same argument as 3(c), we conclude that $G$ is abelian.

(e) If $G = A_4$, determine $A, B$ and the values of $n_2$ and $n_3$.

• Referencing the table for $A_4$ given in chapter 5, note that there are eight distinct 3-cycles, each of which has order 3. Since the subgroup generated by each 3-cycle $\alpha$ contains three elements, including $e, \alpha$, and $\beta$, where $\beta$ is also a 3-cycle, we see that each 3-cycle lies in one of four different 3-Sylow subgroups. Thus, $n_3 = 4$. Following the argument presented in part (b), it must be true that $n_2 = 1$. Note that $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ forms a subgroup of order 4, so it must be the lone 2-Sylow subgroup.

(f) If $G = S_3 \times Z_2$, determine $A, B$ and the values of $n_2$ and $n_3$.

• First note that there is only one 3-Sylow subgroup of $S_3$ (which happens to be $A_3$), which is evident by applying Sylow’s Third Theorem on $S_3$. There are three additional elements of $S_3$ besides those in $A_3$, namely $(12), (13)$, and $(23)$, each having order 2. Thus $S_3$ has an identity, two elements of order 3, and three elements of order 2. $Z_2$ has just two elements: $\epsilon$, and another of order 2. Since there is no order 4 subgroup in $S_3$ or $Z_2$, any order 4 subgroup of $G$ must be formed by order 2 subgroups from both $S_3$ and $Z_2$. These subgroups are precisely $\langle(12)\rangle \times Z_2$, $\langle(13)\rangle \times Z_2$, and $\langle(23)\rangle \times Z_2$. Thus, $n_2 = 3$. Similarly, since 3 is prime and the only order 3 subgroup in $S_3$ or $Z_2$ is $A_3$, the only order 3 subgroup of $G$ is $A_3 \times \{\epsilon\}$. Thus, $n_3 = 1$.

(g) (Bonus) Can you find a group of size 12 such that $A = Z_4$, $n_2 = 3$ and $n_3 = 1$?

(h) (Bonus) Can you find a group of size 12 such that $A = Z_4$, $n_2 = 1$ and $n_3 = 4$?

(i) (Bonus) Determine all groups of size 12 up to isomorphism.

6. (Bonus) Determine all groups of size 18 up to isomorphism.