Complete each of the following exercises.

1. Let $X = \mathbb{R}$ and set $T := \{(a, \infty) \mid a \in \mathbb{R}\} \cup \emptyset \cup \{\mathbb{R}\}$.

(a) Prove that $T$ forms a topology on $\mathbb{R}$. We call this the “right ray topology”.
(b) Is $(\mathbb{R}, T)$ a $T_0$ space? a $T_1$ space? a $T_2$ space?
(c) Is $(\mathbb{R}, T)$ metrizable?
(d) For $\alpha \in \mathbb{R}$, what is $\overline{\{\alpha\}}$? That is, what is the closure of a singleton set?
(e) Consider the sequence $\{1, 2, 3, 4, \ldots\}$. What elements of $\mathbb{R}$ does this sequence converge to in the right ray topology?
(f) Consider the sequence $\{-1, -2, -3, -4, \ldots\}$. What elements of $\mathbb{R}$ does this sequence converge to in the right ray topology?
(g) Consider the sequence $\{0, 0, 0, 0, \ldots\}$. What elements of $\mathbb{R}$ does this sequence converge to in the right ray topology?

2. Let $X$ be any set and put $T := \{X - \{\alpha_1, \ldots, \alpha_n\}\} \cup \emptyset$, that is, $T$ is the collection of sets with finite complement together with the empty set.

(a) Prove that $T$ forms a topology on $X$. We call this the “cofinite topology” on $X$.
(b) Is $(X, T)$ a $T_0$ space? a $T_1$ space? a $T_2$ space?
(c) Is $(X, T)$ metrizable?
(d) For $\alpha \in X$, what is $\overline{\{\alpha\}}$? That is, what is the closure of a singleton set?
(e) Consider an arbitrary sequence $\{x_n\}$. What elements of $X$ does this sequence converge to in the cofinite topology? Does it depend on whether $X$ is infinite or not?

3. Let $(X, T)$ be a topological space and let $Y \subseteq X$. We endow $Y$ with the subspace topology to make it into a topology space.

(a) Let $S \subseteq X$ and write $\overline{S \cap Y}$ for the closure of $S \cap Y$ as a subset of $Y$ under the subspace topology. Prove that $\overline{S \cap Y} \subseteq \overline{S} \cap Y$.
(b) Find an example where $\overline{S \cap Y}$ is strictly smaller than $\overline{S} \cap Y$.
(c) If $U \subseteq Y$ is open in $Y$ (under the subspace topology) is $U$ necessarily open in $X$?

4. Freiwald, Chapter 3: E4(a,b,c,d), E6, E8

5. Let $X$ and $Y$ be two topological spaces and let $f : X \to Y$ be a continuous map.

(a) Prove that if $F \subseteq Y$ is closed, then $f^{-1}(F)$ is closed in $X$.
(b) If $Z$ is another topological space and $g : Y \to Z$ is continuous, prove that the composite $g \circ f : X \to Z$ is continuous.