1. **Counting Orbits.** Let a group $G$ act on a set $X$. The number of orbits of the action is given by

$$\frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|,$$

where $\text{Fix}(g) = \{ x \in X : g \cdot x = x \}$.

*Example:* The dihedral group $D_7$ acts on the set $\Omega_2$ of labeled bracelets with 2 red and 5 green beads. Here the role of $X$ is played by $\Omega_2 = \{ \omega : \{1, 2, \ldots, 7\} \rightarrow \{R, G\} : |\omega^{-1}(R)| = 2 \}$. The number of “really different” bracelets with 2 red beads is the number of orbits. The identity fixes all $\binom{7}{2} = 21$ elements of $\Omega$. For any (non-identity) rotation $R'$ of the bracelet, $|\text{Fix}(R')| = 0$, and for any flip $F_i$, $|\text{Fix}(F_i)| = 3$. So the number of really different bracelets is

$$\frac{1}{14}(21 + 3 \cdot 7) = 3.$$

2. **The cycle index.** Let $G$ act on $X$, so we can think of $G$ as giving permutations of the elements of $X$. Say $|X| = n$. Fix $g \in G$ and say $g$ has $\alpha_1$ cycles of length 1, $\alpha_2$ cycles of length 2, $\ldots$, $\alpha_n$ cycles of length $n$. The cycle index of $g$ is

$$\zeta_g(x_1, x_2, \ldots, x_n) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

The cycle index of $G$ is

$$\zeta_G(x_1, x_2, \ldots, x_n) = \frac{1}{|G|} \sum_{g \in G} \zeta_g(x_1, x_2, \ldots, x_n).$$

*Examples:*

$$\zeta_{C_n}(x_1, \ldots, x_n) = \frac{1}{n} \sum_{d|n} \phi(d)x_d^{n/d}$$

$$\zeta_{D_n}(x_1, \ldots, x_n) = \begin{cases} \frac{1}{2n} \sum_{d|n} \phi(d)x_d^{n/d} + \frac{1}{3}(x_2^{n/2} + x_1^2 x_2^{(n/2)-1}), & n \text{ even} \\ \frac{1}{2n} \sum_{d|n} \phi(d)x_d^{n/d} + \frac{1}{2} x_1 x_2^{(n-1)/2}, & n \text{ odd} \end{cases}$$

The cycle indexes for the Platonic solids are given in Table 27.3.2.

3. **Really Different Colorings with $r$ colors.** Let $G$ act on $X$.

*Theorem:* The number of really different colorings of the elements of $X$ with $r$ colors is

$$\zeta_G(r, r, \ldots, r).$$
Example: The number of ways to color a seven bead bracelet with two colors is

\[ \zeta_{D_7}(2,2,\ldots,2) = \frac{1}{14} \left[ \phi(1)x_7^2 + \phi(7)x_7^1 \right] + \frac{1}{2} x_1 x_2^3 |_{x_1 = x_2 = 2} = \frac{1}{14} [2^7 + 6 \cdot 2] + \frac{1}{2} (2 \cdot 2^3) = 18. \]

4. Generating functions. Let \( G \) act on \( X = \{1,2,\ldots,n\} \) and hence on the set \( \Omega \) of all labeled colorings of the elements of \( X \) with colors \( c_1,\ldots,c_k \). So

\[ \Omega = \{ \omega : \{1,2,\ldots,n\} \to \{c_1,\ldots,c_k\} \} \]

and \( |\Omega| = k^n \). Fix \( \omega \in \Omega \) and say \( \omega \) assigns color \( c_1 \) to \( n_1 \) elements of \( X \) (i.e. \( |\omega^{-1}(c_1)| = n_1 \)), \( c_2 \) to \( n_2 \) elements of \( X \), etc. The indicator of \( \omega \) is \( \text{ind}(\omega) = c_1^{n_1} c_2^{n_2} \cdots c_k^{n_k} \). The generating function of a subset \( A \subset \Omega \) is

\[ U_A(c_1,\ldots,c_k) = \sum_{\omega \in A} \text{ind}(\omega). \]

Important: Say \( |X| = 15 \) and there are 3 colors. The coefficient of e.g. \( c_1^5 c_2^3 c_3^7 \) in \( U_A(c_1,c_2,c_3) \) is the number of elements of \( A \) that color the 15 vertices of \( X \) with 5 vertices color \( c_1 \), 3 vertices color \( c_2 \), 7 vertices color \( c_3 \). So the generating function encodes all the coloring information for a fixed number of colors.

Theorem: Say \( X \) is partitioned up into disjoint subsets \( X_1, X_2,\ldots,X_\ell \). Say \( |X_i| = m_i \). Let \( B \subset \Omega \) be the set of labeled colorings that assign the same color to every element of \( X_1 \), the same color of every element of \( X_2 \), etc. Then

\[ U_B(c_1,\ldots,c_k) = (c_1^{m_1} + \ldots + c_k^{m_1})(c_1^{m_2} + \ldots + c_k^{m_2}) \cdots (c_1^{m_\ell} + \ldots + c_k^{m_\ell}) \]

\[ = \prod_{j=1}^{\ell} (c_1^{m_j} + \ldots + c_k^{m_j}) \]

Example: A subdivision is made up of 4 streets. Street A has 5 houses, Street B has 4 houses, Street C has 4 houses, and Street D has 6 houses. For unknown reasons, all houses on a street must be painted the same color. The available colors are red, blue, green, yellow and white.

(a) How many ways can the houses be painted?

(b) In how many ways can the houses be painted so we end up with exactly 4 red houses and 4 green houses?

(c) In how many ways can the houses be painted so we end up with at least one blue house?

Answers: (a) Each street has 5 color choices, so there are \( 5^4 \) colorings.
(b) There are 19 houses. We get a contribution to the answer from the coefficient of each of the terms in $U_B(R, B, G, Y, W)$ of the form $R^4B^2G^2Y^9W^2$, where $4 + x + 4 + y + z = 19$, or $x + y + z = 11$. Now

$$U_B(R, B, G, Y, W) = (R^5 + B^5 + G^5 + Y^5 + W^5)(R^4 + B^4 + G^4 + Y^4 + W^4)$$

$$\cdot (R^4 + B^4 + G^4 + Y^4 + W^4)(R^6 + B^6 + G^6 + Y^6 + W^6).$$

To get a term of the form $R^4B^2G^4Y^9W^2$, we must pick $R^4$ from the second factor on the right hand side, and $G^4$ from the third factor or vice versa. In the first case, there are 3 choices from the first factor ($B^5$ or $Y^5$ or $W^5$), and 3 choices from the last factor, so we get 9 choices. In the second case, we also get 9 choices, so there are 18 choices in all.

(c) The number of terms in $U_B(R, B, G, Y, W)$ is $5^4$. If a term has no $B$ in it, we must pick one of the four remaining colors $R, G, Y, W$ from each of the four terms. So there are $4^4$ terms with no $B$, which means there are $5^4 - 4^4$ terms with at least one $B$.

5. **Polya’s Theorem.** If $D \subset \Omega$ has one element from each orbit of $G$, then $U_D(c_1, \ldots, c_k)$ keeps track of the number of really different colorings of the elements of $X$. For example, if $|X| = 15$ and there are 3 colors, the coefficient of $R^7G^5Y^3$ is the number of really different colorings with 7 red elements, 5 green elements, 3 yellow elements.

So if we can compute $U_D(c_1, \ldots, c_k)$, we can solve all of our “really different”-type coloring problems. The problem is that $U_D(c_1, \ldots, c_k)$ is a pain to compute. Polya’s Theorem is a remarkable shortcut to compute $U_D(c_1, \ldots, c_k)$.

**Polya’s Theorem:**

$$U_D(c_1, \ldots, c_k) = \zeta_G(x_1, x_2, \ldots, x_n)|_{x_1 = c_1 + \ldots + c_k, x_2 = c_2 + \ldots + c_k, \ldots, x_n = c_n + \ldots + c_k}$$

**Example:** How many really different ways can we color the faces of a cube with 2 faces red, 2 faces green, and 2 faces yellow?

**Answer:** Coloring the faces of the cube is the same as coloring the vertices of the inscribed (or dual) octahedron, which has cycle index

$$\frac{1}{24}(x_1^6 + 6x_1^2x_4 + 3x_1^2x_2^2 + 6x_2^3 + 8x_3^2).$$

So we want the coefficient of the term $R^2G^2Y^2$ in

$$\frac{1}{24}(x_1^6 + 6x_1^2x_4 + 3x_1^2x_2^2 + 6x_2^3 + 8x_3^2)|_{x_1 = R + G + Y, x_2 = R^2 + G^2 + Y^2, x_3 = R^3 + G^3 + Y^3, x_4 = R^4 + G^4 + Y^4}$$

The first term $x_1^6 = (R + G + Y)^6$ contributes $\frac{6!}{2!2!2!} = 90$, the second term contributes 0 (since every term has either an $R$, $G$, or $Y$ with exponent at least 4), the third term
contributes $3 \cdot 3 \cdot 2 \cdot 1 = 18$ (as we can chose either $R^2, G^2$ or $Y^2$ from $x_1^2$, then one of
the remaining two colors squared from the first factor of $x_2$ and the last color squared
from the second $x_2$), the fourth term $6x_2^3$ contributes $6 \cdot 6 = 36$ (since we can pick one
of $R^2, G^2, Y^2$ from the first $x_2$, one of the remaining 2 colors from the second $x_2$, and
the remaining 1 color from the last $x_2$, giving $3 \cdot 2 \cdot 1 = 6$ ways), and the last term
contributes 0. So the answer is $\frac{90 + 18 + 36}{24} = 5$. 