MA 412 – Complex Variables
Exam #2

Name:

Instructions: To receive full credit you must show all work. Explain your
answers fully and clearly. You may refer to theorems/facts in the book or from
class. No calculators, books or notes of any form are allowed. Good luck!

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1. (18 points)

- Define what it means for a function $f(z)$ to be entire.

Several possible def

1. $f$ is entire if $f(z)$ is analytic at every point $z_0 \in \mathbb{C}$

2. $f$ is entire if $f'(z_0)$ exists at every $z_0 \in \mathbb{C}$

- Is the function $f(z) = e^z \sin(2z - 1)$ entire? Explain your reasoning.

Yes. $e^z$ is entire

- $\sin(z)$ is entire and so is $\sin(2z-1)$ since it is the composition of $\sin(z)$ with a polynomial.

- The product of entire functions is entire $\Rightarrow f(z)$ is entire.

- Is the function $f(z) = \sqrt{z} = \exp(\frac{1}{2}\log(z))$ entire? Explain your reasoning.

No. $\log(z)$ is not analytic for $z = x$, $x \leq 0$, so $\exp(\frac{1}{2}\log(z))$ is not analytic there either. It has a branch cut along the negative real axis.

- $\sqrt{z}$ is not entire.
2. (12 points)
Evaluate the following multivalued expressions

- \( \log(-2+2i) \)

\[-2+2i = 2\sqrt{2} \exp\left(\frac{3\pi i}{4}\right) \]

\[\Rightarrow \log(-2+2i) = \ln(2\sqrt{2}) + i\left(\frac{3\pi}{4} + 2n\pi\right) \quad \forall n \in \mathbb{Z} \]

\[-(-i)^4 \]

\[(-i)^4 = e^{i\log(-i)} \quad \log(-i) = -\frac{\pi i}{2} + 2n\pi \quad \forall n \in \mathbb{Z} \]

\[= e^{i\left(-\frac{\pi}{2}i + 2n\pi\right)} \quad \forall n \in \mathbb{Z} \]

\[= e^{-\left(-\frac{\pi}{2} + 2n\pi\right)} \]

\[= e^{\left(\frac{\pi}{2} - 2n\pi\right)} \quad \forall n \in \mathbb{Z} \]

\[\left(\text{since as } e^{\left(\frac{\pi}{2} + 2n\pi\right)} \quad \forall n \in \mathbb{Z}\right) \]
3. (16 points)

Determine the region in which the following functions are analytic, carefully drawing the branch cuts and singularities. Explain your reasoning.

\[
\frac{\log(3-2z)}{z^2 + 16}
\]

Problems occur when \(3-2z \leq 0\) on \(z^2 + 16 = 0\).

\[
3 - 2z \leq 0
\]

\[
\frac{3}{2} \leq z \quad (z \text{ is real } \Rightarrow z \leq \frac{3}{2})
\]

\[
z^2 + 16 = 0 \quad \Rightarrow \quad z = \pm 4i
\]

\[
\sqrt{z^2 + 25},
\]

where the principal branch of the square root is taken.

\[
\sqrt{z^2 + 25} = \exp\left(\frac{1}{2} \log (z^2 + 25)\right)
\]

Problems occur when \(z^2 + 25\) is real \(\Re(z) \leq 0\). This happens when \(z = \alpha i \quad \alpha \in \mathbb{R} \quad |\alpha| > 5\).
4. (10 points)

Compute the contour integral

\[ \int_C zdz \]

where \( C \) is the contour from \(-3i\) to \(3\) along the circle \(|z| = 3\) by parametrizing \( C \) and direct evaluation.

\[
C : z(t) = 3e^{it}, \quad -\frac{\pi}{2} \leq t \leq 0
\]

\[ z'(t) = 3i e^{it} \]

\[
\int_C zdz = \int_{-\frac{\pi}{2}}^{0} 3e^{it} (3i e^{it}) \, dt
\]

\[
= \int_{-\frac{\pi}{2}}^{0} 3e^{-it} 3i e^{it} \, dt
\]

\[
= \int_{-\frac{\pi}{2}}^{0} 9i \, dt = 9i \left( \frac{\pi}{2} \right)
\]

\[
= \frac{9\pi i}{2}
\]
5. (10 points)

Evaluate the contour integral

\[ \int_C \frac{dz}{\sqrt{z}} \]

where \( C \) is the contour from \( z = 1 + i \) to \( 2 + 4i \) along the parabola \( y = x^2 \) and \( \sqrt{z} \) denotes the principal branch. (Hint: find an antiderivative.)

\[ \frac{1}{\sqrt{z}} = e^{\frac{1}{2} \log(z)} \]

which is analytic in a neighborhood of \( C \).

\[ \frac{1}{\sqrt{z}} = \frac{d}{dz} (e \frac{1}{2} 2 \sqrt{z}) \]

So

\[ \int_C \frac{dz}{\sqrt{z}} = 2 \sqrt{z} \bigg|_{1+i}^{2+4i} = 2 (\sqrt{2+4i} - \sqrt{1+i}) \]

This cannot be simplified much more.
6. (10 points)
Show that
\[ \left| \int_{C} \frac{z-1}{z^3 + 2} \, dz \right| \leq \frac{12}{25} \pi \]
where \( C \) is the part of the circle \( |z| = 3 \) from 3 to \(-3\). Clearly show each step in your estimate and which inequalities are being used.

\[ L = \text{length of contour} = 3\pi \]

To find \( M \) such that
\[ \left| \frac{z-1}{z^3 + 2} \right| \leq M \quad \text{on} \ C, \]
use the triangle inequality:
\[ |z-1| \leq |z| + 1| = 4 \quad \text{on} \ C \]
\[ |z^3 + 2| \geq |z^3| - 2| = |z|^3 - 2| = |27-2| = 25 \]

\[ \Rightarrow \frac{1}{|z^3 + 2|} \leq \frac{1}{25} \]

\[ \Rightarrow \left| \frac{z-1}{z^3 + 2} \right| \leq \frac{4}{25} = M. \]

\[ \left| \int_{C} \frac{z-1}{z^3 + 2} \, dz \right| \leq ML = \frac{12\pi}{25}. \]
7. (24 points)

Let

\[ f(z) = \frac{z^3}{(z+2)^2(z-4)}. \]

Evaluate the following contour integrals, in each case explaining your reasoning and referring to the relevant theorems.

(a) \( \int_{C_1} f(z) \, dz \) where \( C_1 \) is the positively oriented circle \( |z-i| = 1 \)

\( f(z) \) is analytic on and inside \( C_1 \), so by the Cauchy-Goursat theorem:

\[ \int_{C_1} f(z) \, dz = 0 \]

(b) \( \int_{C_2} f(z) \, dz \) where \( C_2 \) is the positively oriented square with corners at \(-3-i, -i, 2i, -3+2i\).

By the Cauchy integral formula:

\[ \int_{C_2} f(z) \, dz = \int_{C_2} \frac{g(z)}{(z+2)^2} \, dz \]

\[ g(z) = \frac{z^3}{z-4} = \frac{z^3}{(z-4)(z+2)} \]

Since \( g(z) \) is analytic in \( C_2 \),

\[ g'(z) = \frac{(z-4)3z^2 - z^3}{(z+2)^2} = \frac{2z^3 - 12z^2}{(z+2)^2} \]

\[ g'(-2) = \frac{2(-2)^3 - 12(-2)^2}{(-6)^2} = \frac{-2(-2)^3 - 12(-2)^2}{(-6)^2} = \frac{-2 \cdot 16}{36} = -\frac{16}{9} \ln 4 \]

\[ = 2\pi i \cdot \frac{16}{9} \]

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(c) \( \int_{C_3} f(z) \, dz \) where \( C_3 \) is the negatively oriented circle \(|z - 5| = 2\).

\[
\int_{C_3} f(z) \, dz = \int_{C_3} \frac{h(z)}{z - 4} \, dz
\]

where \( h(z) = \frac{z^3}{(z+2)^2} \) and \( h(z) \) is analytic in \( C_3 \).

By the Cauchy integral formula,
\[
\int_{C_3} \frac{h(z)}{z - 4} \, dz = -2\pi i \, h(4) \quad \text{because } C_3 \text{ is a negatively oriented circle.}
\]

\[
= -2\pi i \left( \frac{4^3}{6^2} \right) = -2\pi i \cdot \frac{64}{36} = -\frac{16\pi i}{9}
\]

(d) \( \int_{C_4} f(z) \, dz \) where \( C_4 \) is the positively oriented circle \(|z| = 8\). (Hint: how does this integral relate to those over \( C_2 \) and \( C_3 \)?)

\( f(z) \) is analytic in the shaded region.

By the Cauchy integral formula,
\[
\int_{C_4} f(z) \, dz = \int_{C_2} f(z) \, dz + \int_{C_3} f(z) \, dz
\]

\( = \int_{C_2} f(z) \, dz - \int_{C_3} f(z) \, dz \)

\( = 0 \)