Repeated eigenvalues

Sometimes the characteristic polynomial has the same real root twice. When this happens, we say that the eigenvalues are “repeated.”

Example. \( \frac{dY}{dt} = AY \) where \( A = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix} \).

The characteristic polynomial of \( A \) is \((\lambda - 3)^2\), so there is only one eigenvalue, \( \lambda = 3 \). Let’s calculate the associated eigenvectors:

But we already know how to solve this system. How?
We obtain the general solution

\[ Y(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 e^{3t} + 2y_0 t e^{3t} \\ y_0 e^{3t} \end{pmatrix} = e^{3t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t e^{3t} \begin{pmatrix} 2y_0 \\ 0 \end{pmatrix}. \]

Note that this general solution is not written as a linear combination. Every nontrivial solution contains the first term, and most solutions contain both terms.

We use this result to motivate a different technique that we use to solve systems with repeated eigenvalues. We use a guessing technique where we guess a solution of the form

\[ Y(t) = e^{\lambda t} V_0 + te^{\lambda t} V_1. \]

Note that the initial condition for this solution is \( V_0 \).
Fact from linear algebra: If $A$ is a $2 \times 2$ matrix with a repeated eigenvalue $\lambda$ and $V_0$ is any vector, then either

1. $(A - \lambda I)V_0 = 0$ (in other words, $V_0$ is an eigenvector), or
2. the vector $V_1 = (A - \lambda I)V_0$ is an eigenvector of $A$.

Example. $\frac{dY}{dt} = AY$ where

$$A = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix}.$$ 

The characteristic polynomial of $A$ is $\lambda^2 + 4\lambda + 4$, so $\lambda = -2$ is a repeated eigenvalue.
What is the long-term behavior of a system with a repeated, negative eigenvalue?

It is interesting to look at this example using two of the tools on the CD. Using `LinearPhasePortraits`, we can see that this system is on the boundary between spiral sinks and real sinks.

We can also use `HPGSystemSolver` to plot the phase portrait and a typical pair of $x(t)$- and $y(t)$-graphs.
Unusual case of repeated eigenvalues: There is one type of linear system that has repeated eigenvalues that is different from the examples we have discussed.

**Example.** Consider $\frac{dY}{dt} = AY$ where $A$ is the diagonal matrix

$$
\begin{pmatrix}
\lambda & 0 \\
0 & \lambda
\end{pmatrix}.
$$

What are its eigenvalues and eigenvectors?
Finally consider the example

\[ \frac{dY}{dt} = \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} Y. \]

Its characteristic polynomial is \( \lambda^2 + 3\lambda \). So its eigenvalues are \( \lambda = -3 \) and \( \lambda = 0 \). (If a system has 0 as an eigenvalue, we say that it is degenerate. The matrix \( A \) of coefficients is singular—see your class notes for March 1 and March 3.)