The Laplace transform

For the remainder of the semester, we are going to take a somewhat different approach to the solution of differential equations. We are going to study a way of transforming differential equations into algebraic equations.

We begin with a little review of improper integrals.

**Example.** Consider the improper integral

\[ \int_{0}^{\infty} e^{-2t} \, dt. \]
**Example.** Consider the improper integrals

$$\int_0^\infty e^{-st} \, dt$$

for various values of $s$.

**Definition.** The *Laplace transform* of the function $y(t)$ is the function

$$Y(s) = \int_0^\infty y(t) e^{-st} \, dt.$$ 

This transform is an “operator” (a function on functions). It transforms the function $y(t)$ into the function $Y(s)$. 

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Notation: We often represent this operator using the script letter \( \mathcal{L} \). In other words,

\[
\mathcal{L}[y] = Y.
\]

For example, \( \mathcal{L}[1] = \frac{1}{s} \).

Note that, even if \( y(t) \) is defined for all \( t \), the Laplace transform \( Y(s) \) may not be defined for all \( s \).

**Example.** Let’s compute \( \mathcal{L}[e^{at}] \) using the definition and the improper integrals we have already computed:

**Examples.** Using Mathematica to calculate the improper integrals, we see that:

\[
\mathcal{L}[\sin t] = \frac{1}{s^2 + 1} \quad \text{for} \quad s > 0
\]

\[
\mathcal{L}[e^{2t} \sin 3t] = \frac{3}{s^2 - 4s + 13} \quad \text{for} \quad s > 2
\]

\[
\mathcal{L}[t^4] = \frac{24}{s^5} \quad \text{for} \quad s > 0
\]

\[
\mathcal{L}[\sin 2t] = \frac{2}{s^2 + 4} \quad \text{for} \quad s > 0,
\]

\[
\mathcal{L}[t \cos \sqrt{2} t] = \frac{s^2 - 2}{(s^2 + 2)^2} \quad \text{for} \quad s > 0
\]

\[
\mathcal{L}[e^{i\omega t}] = \frac{1}{s - i\omega} \quad \text{for} \quad s > 0
\]
Properties of the Laplace transform  There are two properties of the Laplace transform that make it well suited for solving linear differential equations:

1. \( \mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0) \)

2. \( \mathcal{L} \) is a linear transform

Both of these properties are extremely important, but the surprising one is #1. Let’s consider

\[
\mathcal{L}\left[\frac{dy}{dt}\right] = \int_{0}^{\infty} \left( \frac{dy}{dt} \right) e^{-st} \, dt.
\]

In fact, before we consider the improper integral, let’s apply the method of integration by parts to the indefinite integral

\[
\int \left( \frac{dy}{dt} \right) e^{-st} \, dt.
\]
Now let’s see how we can use the Laplace transform to solve an initial-value problem.

**Example.** Solve the IVP

\[
\frac{dy}{dt} - 3y = e^{2t}, \quad y(0) = 4.
\]

1. Transform both sides of the equation:

2. Solve for \( \mathcal{L}[y] \):

3. Calculate the inverse Laplace transform:

Is this the right answer? Do we need Laplace transforms to calculate it?