1. (20 points)
   (a) Solve the following initial value problem.

   \[
   \frac{d^2 y}{dt^2} + 4y = -2e^{-t}
   \]

   \[y(0) = 0, \ y'(0) = 0\]

   \[\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i\]

   \[y_h(t) = k_1 \cos 2t + k_2 \sin 2t\]

   \[y_f(t) = Ce^{-t}\]

   \[
   \frac{d^2 y_f}{dt^2} + 4y_f = 5Ce^{-t} = -2e^{-t}
   \]

   \[\therefore \; C = -\frac{2}{5}\]

   \[y(t) = y_h(t) + y_f(t) = k_1 \cos 2t + k_2 \sin 2t - \frac{2}{5}e^{-t}\]

   \[y(0) = y'(0) = 0 \Rightarrow \; k_1 = \frac{2}{5}, \; k_2 = -\frac{1}{5}\]

   \[\therefore \; y(t) = \frac{2}{5} \cos 2t - \frac{1}{5} \sin 2t - \frac{2}{5}e^{-t}\]
1. (20 points, continued)
   
   (b) Provide a rough sketch of your solution and describe the behavior of your solution as $t$ goes to positive infinity.

   $y = -\frac{2}{5}e^{-t}$

   As $t$ goes to infinity, $-\frac{2}{5}e^{-t}$ will be smaller.

   So $y(t) \approx \frac{2}{5} \cos 2t - \frac{1}{5} \sin 2t$

   $= \sqrt{\frac{1}{5}} \cos (2t - \phi)$.
2. (20 points)
(a) Find the general solution of the following equation.

\[
\frac{d^2y}{dt^2} + 16 \frac{dy}{dt} = 2\cos(2t)
\]

\[
\lambda^2 + 16 = 0
\]
\[
\lambda = \pm 4i
\]

\[
y_h(t) = k_1 \cos(4t) + k_2 \sin(4t)
\]

\[
2\cos(2t) = \text{real part of } 2e^{2it}
\]

\[
y_p(t) = \text{real part of } C e^{2it}
\]

\[
-4Ce^{2it} + 16Ce^{2it} = 2e^{2it}
\]

\[
C = \frac{2}{12} = \frac{1}{6}
\]

\[
y_p(t) = \frac{1}{6} \cos(2t)
\]

\[
y(t) = y_h(t) + y_p(t)
\]

\[
y(t) = k_1 \cos(4t) + k_2 \sin(4t) + \frac{1}{6} \cos(2t)
\]
2. (20 points, continued)
   (b) Determine the frequency of the beats and the rapid oscillations of your solution to part (a).

   \[ \text{frequency of the beats: } \frac{4 - 2}{2 \cdot 2\pi} = \frac{1}{2\pi} \]

   \[ \text{"of the rapid oscillations:" } \frac{4 + 2}{2 \cdot 2\pi} = \frac{3}{2\pi} \]

   (c) What could you change the forcing term (right hand side of the equation in part (a)) to make the forcing resonant. **You must justify your answer to receive any credit.**

   The resonance occurs when the (angular) frequencies are the same. ∨

   Since the natural (angular) frequency is 4, if we replace \(2\cos(2t)\) by \(2\cos(4t)\), then the resonance occurs.
3. (20 points)
Solve the following initial value problem.
\[
\frac{d^2 y}{dt^2} + 4 \frac{dy}{dt} + 40 = 0
\]
\[y(0) = 1, y'(0) = 8\]
\[
\frac{d^2 y}{dt^2} + 4 \frac{dy}{dt} = -40
\]
\[\lambda^2 + 4\lambda = 0 \quad \lambda = 0, -4
\]
\[y_h(t) = k_1 + k_2 e^{-4t}
\]
\[y_p(t) = -10t
\]
\[
y(t) = y_h(t) + y_p(t) = k_1 + k_2 e^{-4t} - 10t
\]
\[y(0) = k_1 + k_2 = 1
\]
\[y'(t) = -4k_2 e^{-4t} - 10
\]
\[y'(0) = -4k_2 - 10 = 8
\]
\[
\therefore k_2 = -\frac{18}{4} = -\frac{9}{2}
\]
\[k_1 = 1 - k_2 = \frac{11}{2}
\]
\[
\therefore y(t) = \frac{11}{2} - \frac{9}{2} e^{-4t} - 10t
\]
4. (20 points)
   (a) Determine the equilibrium points of the following system. Restrict your analysis to
   the first quadrant $\{x \geq 0, y \geq 0\}$.

   \[
   \frac{dx}{dt} = x(2-x-y) \\
   \frac{dy}{dt} = y(y-x^2)
   \]

   \[\begin{align*}
   \frac{dx}{dt} &= 0 & \text{and} & \frac{dy}{dt} &= 0 \\
   \frac{dx}{dt} &= x(2-x-y) = 0 \\
   \text{either} & \quad x = 0 \quad \text{or} \quad 2-x-y = 0
   \end{align*}\]

   \underline{case \quad x = 0}

   \[\frac{dy}{dt} = y(y-x^2) = y^2 = 0 \quad \therefore \quad y = 0.\]

   \underline{case \quad 2-x-y = 0 \quad \text{i.e.} \quad y = 2-x.}

   \[\frac{dy}{dt} = (y-x^2) = (2-x)(2-x-x^2) = (2-x)(1-x)(2+x)\]

   \[x = 2, 1, -2.\]

   We restricted to \(\{x \geq 0, y \geq 0\}\), thus \(x \neq -2\).

   So \((0,0), (2,0), (1,1)\).
4. (20 points, continued)

(b) Determine the type of equilibrium point for each of the equilibrium points you calculated in (a).

\[ J = \begin{pmatrix} 2 - 2\alpha - \beta & -\alpha \\ -2\alpha y & 2\gamma - \beta^2 \end{pmatrix} \]

at \((0,0)\), \( J = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \)

\( \lambda = 2, 0 \)

"source" (with a zero eigenvalue)

at \((2,0)\), \( J = \begin{pmatrix} -2 & -2 \\ 0 & -4 \end{pmatrix} \)

\( \lambda = -2, -4 \)

Sink

at \((1,1)\), \( J = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \)

\( \lambda^2 - 3 = 0 \Rightarrow \lambda = \pm \sqrt{3} \)

Saddle
5. (20 points)

Solve the following initial value problem using Laplace transforms.

\[ \frac{dy}{dt} + 4y = 1, \quad y(0) = 0 \]

\[
\mathcal{L} \left( \frac{dy}{dt} + 4y \right) = \mathcal{L} (1) \\
\mathcal{L} \left( \frac{dy}{dt} \right) + 4 \mathcal{L} (y) = \mathcal{L} (1) \\
5 \mathcal{L} (y) - y(0) + 4 \mathcal{L} (y) = \mathcal{L} (1) \\
(5 + 4) \mathcal{L} (y) = \frac{1}{5} \\
\mathcal{L} (y) = \frac{1}{5 (5+4)} = \frac{1}{4} \left( \frac{1}{5} - \frac{1}{5+4} \right) \]

\[ y = e^{-4t} \left( \frac{1}{4} \left( \frac{1}{5} - \frac{1}{5+4} \right) \right) \]

\[ = \frac{1}{4} \left( 1 - e^{-4t} \right) \]