Algebras and varieties related to finite subgroups of $\text{Sp}(2n)$

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Abstract

This talk is an introduction to my recent work with Victor Ginzburg.

The Cherednik algebra $H_n$ is the algebra over $\mathbb{C}$ generated by the symmetric group $S_n$ and two sets of generators $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$, where the defining relations are the obvious commutation relations between $S_n$ and $x_i, y_i$, and the relations:

$$[x_i, x_j] = [y_i, y_j] = 0,$$
$$[x_i, y_j] = -s_{ij}, i \neq j$$
$$[x_i, y_i] = s_{i1} + ... + s_{in}.$$ (where $s_{ij}$ is the permutation of $i$ and $j$). This algebra has a filtration given by $\text{deg}(S_n) = 0$, $\text{deg}(x_i) = \text{deg}(y_i) = 1$, and it is known from Cherednik’s work that $\text{gr}(H_n)$ is the smash product $\mathbb{C}[S_n] \bullet \mathbb{C}[x, y]$ (the Poincare-Birkhoff-Witt theorem).

Ginzburg and I proved the following:

**Theorem.** 1. Let $Z_n$ be the center of $H_n$. Then $\text{gr}(Z_n) = \mathbb{C}[x, y]^{S_n}$

2. Let $M_n = \text{Spec}(Z_n)$. Then $M_n$ is a smooth, affine, symplectic algebraic variety of dimension $2n$.

3. There exists an algebraic vector bundle $V$ on $M_n$ of dimension $n!$ such that $H_n = \text{End}(V)$. The group $S_n \subset H_n$ acts on fibers of this bundle as in the regular representation. In particular, all irreducible representations of $H_n$ are of dimension $n!$ and are parametrized by points of $M_n$. 

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4. The variety $M_n$ is isomorphic to the Kazhdan-Kostant-Sternberg-Wilson "Calogero-Moser space", which is the set of pairs of n by n matrices $X, Y$ such that $[X, Y] + 1$ has rank 1, modulo conjugation.

Some, but not all, of these results can be generalized to the case when $S_n$ is replaced by any Coxeter group and even any group generated by symplectic reflections: some as theorems, some as conjectures.

I will try to describe some of these results and also their quantum analogs.