Instructions: Please do 4 out of 5 problems in Part I, 3 out of 6 problems in Part II, and 1 out of 3 problems in Part III. Partial credit will be awarded, so please indicate clearly which problems you want graded.

Part I: Do 4 out of 5 problems.

1. Compute \( \frac{d}{dx} \int_x^e t^2 + 3 \log(t) \, dt \).

2. Compute \( \lim_{t \to \infty} \int_0^1 e^{-tx} \, dx \). Justify all your steps.

3. Compute the line integral \( \int_C (x^4 + \cos(x) + y) \, dx + (y^5 + \sin(y) + x) \, dy \), where \( C \) is the upper half of the circle of radius 2 centered at the origin and travelled counterclockwise.

4. Give an \( \epsilon-\delta \) proof of the continuity of \( f(x) = \sqrt{x} \) at \( x = 0 \).

5. Let \( f : \mathbb{R} \to \mathbb{R} \) be \( C^\infty \). Consider the differential equation
   \[
   \frac{dy}{dt} = f(y), \quad y(0) = y_0.
   \]
   For \( \Delta t \) small, compute \( y(\Delta t) \) using one step of Euler's method. Find the error in the one-step Euler method by comparing your answer to the exact solution of \( y(\Delta t) \).

Part II: Do 3 out of 6 problems.

6. Compute \( \lim_{t \to \infty} \int_0^1 e^{-tx} \, dx \). Justify all your steps.

7. Show that \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \) by computing the Fourier series of \( f(x) = x \) on \([-\pi, \pi]\).

8. Prove or give a counterexample: the vector space of all real valued continuous functions on \([0, 1]\) is a complete metric space with respect to the \( L^2 \) metric
   \[
   d(f, g) = \left[ \int_0^1 (f(x) - g(x))^2 \, dx \right]^{1/2}.
   \]

9. What type of fixed point does the system
   \[
   \begin{align*}
   \frac{dx}{dt} &= 5y \\
   \frac{dy}{dt} &= -x - 6y
   \end{align*}
   \]
   have at the origin? Sketch the phase portrait and write down the general solution.
10. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (e^x \cos(y), e^x \sin(y))$. Show that $f$ is locally one-to-one, but is not one-to-one on $\mathbb{R}^2$.

11. Evaluate $\lim_{n \to \infty} \frac{1}{n} \left( \sin(\pi/n) + \sin(2\pi/n) + \ldots + \sin(n\pi/n) \right)$.

**Part III: Do 1 out of 3 problems.**

12. Prove or give a counterexample: If $W$ is a proper subspace of an inner product space $V$, then there exists a vector $\alpha \in V, \alpha \neq 0$, such that $\alpha \perp w$ for all $w \in W$.

13.
   a. Prove that every solution of $\frac{dy}{dt} = \sin(y)$ is bounded above and below for all $t$.
   b. How must you modify your proof for part (a) to show that every solution of $\frac{dy}{dt} = 1 + \sin(y)$ is bounded above and below for all $t$?
   c. How must you modify your proof for part (a) to show that every solution of $\frac{dy}{dt} = t^3 \sin(y)$ is bounded above and below for all $t$?

14.
   a. Let $X$ be a metric space and let $A \subset X$. For $x \in X$, define $d(x, A)$, the distance from $x$ to $A$, to be $d(x, A) = \inf_{y \in A} \{ d(x, y) \}$.
      Show that if $A$ is compact, then there exists $y \in A$ with $d(x, A) = d(x, y)$.
   b. Show that the conclusion of part a. fails if $X$ is complete, but $A$ is only assumed to be closed.
   c. Show that the conclusion of part a. holds if $A$ is closed and $X = \mathbb{R}^n$. 
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Afternoon Exam

PART I. Solve 3 out of the next 4 questions.

1. Find a basis for the space of solutions to the system of equations
\[ \begin{align*}
x_1 - x_2 + 2x_3 + x_4 &= 0 \\
2x_1 + x_2 - 3x_3 + 2x_4 &= 0.
\end{align*} \]

2. Find the eigenvalues of the matrix \( A = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \).

3. Find the greatest common divisor of the polynomials \( f(x) = 12 - x - 13x^2 + x^3 + x^4 \) and \( g(x) = -6 - 7x + x^3 \).

4. Let \( n \) be an odd positive integer. Show that every \( n \times n \) real matrix \( A \) has a nonzero eigenvector in \( \mathbb{R}^n \). Show that this statement is false for even integers \( n \).

PART II. Solve 3 out of the next 5 questions.

1. Find the smallest integer solution to the system of congruences
\[ \begin{align*}
x &\equiv 3 \pmod{13} \\
x &\equiv 5 \pmod{27}.
\end{align*} \]

2. State and prove the Cauchy-Schwarz inequality in \( \mathbb{R}^n \). (Do not assume the triangle inequality).

3. How many subgroups does the group \( G = \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/21\mathbb{Z} \) have? Prove your claim.

4. Let \( U, V \) be subspaces of a finite dimensional real vector space \( W \). Prove: \( \dim(U + V) + \dim(U \cap V) = \dim(U) + \dim(V) \).

5. Find the determinant of the matrix
\[ \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 4 & 8 \\
1 & 3 & 9 & 27 \\
1 & 4 & 16 & 64
\end{pmatrix} \]

PART III. Solve 2 out of the next 4 questions.

1. Consider the real matrix \( A = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} \). Find the complex eigenvalues and eigenvectors of \( A \).
   (Hint: write \( A = aI + bJ + cJ^2 \) for a suitable matrix \( J \), and diagonalize \( J \).)

2. Let \( f \in \mathbb{Q}[x] \) be an irreducible polynomial. Prove that \( f \) has no multiple roots in \( \mathbb{C} \).

3. Consider the additive group \( G = \mathbb{Z}^2 \) of lattice points in the plane. Is this group generated by the elements \( (17, 15) \) and \( (26, 23) \)? Give a complete proof.

4. Let \( V \) be a real vector space. A function \( f : V \to V \) is called an affine transformation if there is a linear transformation \( L : V \to V \) and an element \( b \in V \) such that \( \forall v \in V, f(v) = Lv + b \).
   (a) Show that an affine transformation \( f \) has an inverse if and only if \( L \) is invertible.
   (b) Show that the set \( A \) of invertible affine transformations forms a group.
   (c) Show that if \( G \subset A \) is a finite subgroup, then it has a fixed point \( x \in V \), namely: \( \forall g \in G, g(x) = x \).